

The discretized harmonic oscillator: Mathieu functions and a new class of generalized Hermite polynomials

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We present a general, asymptotical solution for the discretized harmonic oscillator. The corresponding Schrödinger equation is canonically conjugate to the Mathieu differential equation, the Schrödinger equation of the quantum pendulum. Thus, in addition to giving an explicit solution for the Hamiltonian of an isolated Josephson junction or a superconducting single-electron transistor (SSET), we obtain an asymptotical representation of Mathieu functions. We solve the discretized harmonic oscillator by transforming the infinite-dimensional matrix-eigenvalue problem into an infinite set of algebraic equations which are later shown to be satisfied by the obtained solution. The proposed ansatz defines a new class of generalized Hermite polynomials which are explicit functions of the coupling parameter and tend to ordinary Hermite polynomials in the limit of vanishing coupling constant. The polynomials become orthogonal as parts of the eigenvectors of a Hermitian matrix and, consequently, the exponential part of the solution can not be excluded. We have conjectured the general structure of the solution, both with respect to the quantum number and the order of the expansion. An explicit proof is given for the three leading orders of the asymptotical solution and we sketch a proof for the asymptotical convergence of eigenvectors with respect to norm. From a more practical point of view, we can estimate the required effort for improving the known solution and the accuracy of the eigenvectors. The applied method can be generalized in order to accommodate several variables. © 2003 American Institute of Physics. [DOI: 10.1063/1.1561156]

I. INTRODUCTION

This paper is closely related to one of the famous eigenvalue problems, namely that of a one-dimensional harmonic oscillator. It is common knowledge that if the eigenvectors are required to have continuous second-order derivatives, each eigenvector is expressible as a product of a Hermite polynomial and an exponential term. The corresponding eigenvalues are equidistantly spaced and bounded from below. Another way to state the problem is given by the annihilation and creation operators which directly diagonalize the Hamiltonian. In comparison, the quartic anharmonic oscillator was solved by Bender and Wu in Ref. 1. A method for finding eigenvalues for anharmonic oscillators was created by Meißner and Steinborn in Ref. 2. A general method for polynomial potentials was introduced recently by Meurice.^{3,4}

Instead of continuous functions, we consider functions defined only on a discrete, equidistantly-spaced and countable set on \mathbb{R} . The obvious advantage of this approach is that it transforms the problem into an eigenvalue problem of an infinite-dimensional, tri-diagonal matrix. The corresponding Schrödinger equation is canonically conjugate to the Mathieu differential equation.⁵ Numerical solutions for noninteger orders are naturally obtained by diagonalizing the very same matrix, see Ref. 6 and the references therein for applications.

In physics, the discretized harmonic oscillator is manifestly realized by the Hamiltonian of an

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isolated Josephson junction^{7,8} and the Hamiltonian of the, slightly misleadingly named, superconducting single-electron transistor (SSET).^{8,9} Presently, excited states are seldom considered because of the radical approximations under which the Hamiltonian is solved. Even if the excited states are numerically obtained, it is not immediately evident, what happens when the coupling is changed. In this paper we give an explicit, asymptotical solution for the discretized harmonic oscillator which corresponds to strong Josephson coupling in case of the SSET. The same Hamiltonian also describes the so-called quantum pendulum, or a particle in a periodic potential.^{10,11}

The corresponding asymptotical eigenvalues have been available for almost 50 years due to the work of Meixner and Schäfke on Mathieu functions in Ref. 12. First, by calculating the determinant of the matrix representation accurately enough, we continue this expansion by several orders in the coupling parameter. Then we propose an ansatz that transforms the matrix equation into an infinite set of algebraic equations and proceed by recursively solving these equations. The general properties of the coefficients in the ansatz can be obtained by studying any occurring regularities and reinserting these into the solution. Thus, in addition to the eigenvalues, we have successfully conjectured the general form of the asymptotical eigenvectors. In each order of the expansion, the expressions are quoted in terms of an arbitrary quantum number, n , whenever possible. The leading terms have been determined and rephrased in terms of an arbitrary order, m , too. We find that the eigenvectors are asymptotical solutions of certain differential equations, which enables us to obtain further orders in their expansions.

The only real-valued parameter in the solution is the coupling constant, because all coefficients, both in the eigenvalues and in the ansatz are rational numbers. As a practical application, the rate of convergence of the solution towards numerically obtained, “exact,” solution, can be reliably estimated. In the asymptotical limit, the dependence in terms of n and m assumes the form of a simple monomial, at least down to the limits of numerical precision.

The solutions of order $m \leq 5$ are very simple to program and directly apply as numerical solutions of the discretized harmonic oscillator. For sufficiently small values of the coupling constant the eigenvectors are practically exact and thus they facilitate studies which require the structure of the excited states. We have proven, with the help of recursion relations of Hermite polynomials, that the first three leading orders of the obtained solution are correct. The calculation up to the seventh order should be performed in the future. We also outline an explicit proof concerning the normwise convergence of the eigenvectors. The asymptotical nature of the solutions must be stressed. A very thorough introduction on the subject has been given by Boyd in Ref. 13.

It is justified to ask, is the proposed solution completely new. The answer is, naturally, yes and no. Both discretized and discrete harmonic oscillators have been widely studied before. Both cases are related to orthogonal polynomials, so the work of Kravchuk¹⁴ and Hahn¹⁵ must be mentioned. The discrete harmonic oscillator, where the position coordinate is restricted to a finite number of values, is explicitly solved by Kravchuk polynomials as shown by Lorente in Ref. 16. Several discretizations of the harmonic oscillator have been previously solved, each giving rise to a specific class of generalized Hermite polynomials. Discretization by an exponential lattice $\{-q^n, q^n | n \in \mathbb{Z}\}$, where $0 < q < 1$, defines the so-called q -deformed harmonic oscillator and generalized q -Hermite polynomials which are rigorously discussed by Berg and Ruffing in Ref. 17. For other applications of the q -deformed harmonic oscillators, see, e.g., Refs. 18 and 19, where other discretizations are reviewed, too. Borzov, in Ref. 20, considers generalized derivation operators as generators of Hermite polynomials and states that the generalized Hermite polynomials either satisfy a second-order differential operator or there is no differential equation of finite order for these polynomials. Many other types of generalizations are also known, see e.g., the multidimensional Hermite polynomials of Rösler,²¹ Hermite polynomials orthogonal with respect to the measure $|\xi|^\gamma \exp(-\xi^2) d\xi$, where $\gamma > -1$,^{22,23} and parabosonic Hermite polynomials.²⁴ In the future, it must be established whether the presented class of Hermite polynomials is related to the q -Hermite polynomials, if it results from some other discretization or is it an explicit example of the second group of Borzov’s categorization. Complementary results concerning the introduction of distant boundaries for the continuous problem are also known.^{25,26} Finally, it should be empha-

sized that instead of deforming the harmonic oscillator, we solve its common-sense discretization, used especially in numerical calculations. The asymptotical effects of the discretization are explicitly calculated.

We also briefly consider the abruptly changing nature of the solutions when the coupling constant vanishes. This behavior is evident for both versions of the harmonic oscillator and the Mathieu differential equation. The asymptotical nature of the solutions and the eigenvalues is caused by this divergence. For the Mathieu equation this has been well documented, see, e.g., Refs. 5, 12, and 27. A more physically motivated approach is given by Bender, Pelster, and Weissbach in Ref. 28, where, e.g., the instanton equation and the Blasius equation are examined. The present methods are closely related to these, although we can not carry the calculation as far in the perturbative expansion. This is explained by the necessity of obtaining the expansion for the eigenvalues which makes the present problem technically more demanding.

The present method can be generalized in a fairly obvious manner. Other differential equations with analytical solutions can be discretized in the same manner if the correct expansions are found for all parts of the solution. An easier generalization is related to multi-dimensional difference equations with harmonic (quadratic) potential terms. The existing solution²⁹ for Hamiltonians of one-dimensional arrays of Josephson junctions become more transparent with the help of present formalism.

The paper is organized as follows. In Sec. II we define the discretized harmonic oscillator and connect it to the Mathieu differential equation as well as the continuous case. The solution ansatz and the resulting set of equations are reviewed. In Sec. III we quote our conjectures for the general form of the coefficients in the ansatz. We also present the explicit values of the leading coefficients. In Sec. IV we study solving the set of equations which yields the asymptotical eigenvectors. Efficient truncations of the set of equations are explained. The effort for improving the obtained results with the present method is estimated. In Sec. V we prove that the solution satisfies the difference equations, at least for the three leading orders. The rate of convergence and the induced asymptotical orthonormality are also reviewed. Finally, in Sec. VI the conclusions are drawn and an outlook of future possibilities is given.

A final note for those that are only interested in applying these results in numerical and/or theoretical analysis. Please review the beginning of Sec. II in order to find the correct parameters for the discretized harmonic or Mathieu equation. Then proceed to Sec. III and use the given expressions as approximate solutions in Eq. (30).

II. THE DISCRETIZED HARMONIC OSCILLATOR

The eigenvalue problem corresponding to the harmonic oscillator is the differential equation for $\psi(x)$,

$$-\frac{1}{2} \frac{d^2\psi}{dx^2} + \frac{\omega^2 x^2}{2} \psi = \lambda \psi. \tag{1}$$

The eigenvectors corresponding to the well-known eigenvalues,

$$\lambda_n = \omega(n + 1/2), \tag{2}$$

where $n=0,1,2,\dots$, are given by

$$\psi_n(x) = A_n H_n(\xi) e^{-\xi^2/2}. \tag{3}$$

Here $\xi = \sqrt{\omega} x$, A_n is a normalization factor, and H_n is the Hermite polynomial of order n . The Hermite polynomials are solutions of the Hermite differential equation

$$y'' - 2xy' + 2ny = 0, \tag{4}$$

where $n = 0, 1, 2, \dots$. For our convenience, we write the polynomials, given by Rodrigues' formula, as

$$H_n(\xi) = (-1)^n \exp(\xi^2) \frac{d^n}{d\xi^n} \exp(-\xi^2) = \sum_{k=0}^{k'} h_k^{(n)} \xi^{n+2(k-k')}, \tag{5}$$

where $k' := \lfloor n/2 \rfloor$, i.e., $k' = n/2$ if n is even and $k' = (n-1)/2$ if n is odd. The quantity k' proves to be extremely useful in further analysis. The Hermite polynomials satisfy the recursion relation

$$H_{n+1}(\xi) = 2\xi H_n(\xi) - 2n H_{n-1}(\xi). \tag{6}$$

Many of the generalizations of the Hermite polynomials boil down to a generalization of this recursion relation.^{16,17,19,24}

The discretized version of Eq. (1) is obtained by restricting the values of x onto an evenly spaced, countable subset of \mathbb{R} . This corresponds e.g., to the discretization of charge in case of a Josephson junction or a SSET. Only the constant nearest-neighbor coupling is retained which yields a tri-diagonal matrix $H(x_0)$ with nonzero matrix elements

$$H_{jj}(x_0) = \frac{1}{2}\omega^2(j-x_0)^2, \quad H_{j+1,j}(x_0) = H_{j,j+1}(x_0) = -\frac{1}{2}. \tag{7}$$

Here the parameter $x_0 \in [-\frac{1}{2}, \frac{1}{2}]$ is the displacement of the origin with respect to the matrix element $j=0$. All eigenvalues of $H(x_0)$ have been translated by -1 in order to simplify the diagonal matrix elements. The standard way to write the Hamiltonian of an inhomogeneous SSET is obtained from Eqs. (7.36) and (7.39) of Ref. 8 and rephrasing it in terms of the number operator for Cooper pairs yields the matrix

$$H_{jj}^{(\text{SSET})}(N_0) = E_C(j-N_0)^2, \quad H_{j+1,j}^{(\text{SSET})}(N_0) = H_{j,j+1}^{(\text{SSET})}(x_0) = -\frac{1}{2}E_J(\theta), \tag{8}$$

where N_0 is the number of Cooper pairs which minimizes the charging energy, $E_C = (2e)^2/2C_\Sigma$ is the unit of charging energy, and $E_J(\theta)$ is the effective Josephson energy which depends on the total phase θ across the SSET. Consequently, we solve the Hamiltonian of SSET if we find the eigenenergies and eigenvector for the discretized harmonic oscillator with $\omega = (2E_C/E_J(\theta))^{1/2}$.

In the following, we are searching for eigenvectors with finite Euclidean norm, i.e.,

$$\|\psi\|^2 = \sum_{j=-\infty}^{\infty} |\psi_j|^2 < \infty. \tag{9}$$

The existence and uniqueness of such solutions follows from the generalization of the Gershgorin eigenvalue theory by Shivakumar, Rudraiah, and Williams in Ref. 30. First the number of eigenvalues of $H(x_0)$ on a given interval can be shown to coincide with number of eigenvalues for a finite-dimensional truncation of the matrix, $H^{(N)}(x_0)$, if the dimension N is sufficiently large. A sufficient condition for this is that, in the ordered sequence of diagonal matrix elements, the difference between two consecutive values exceeds $4 \times |-1/2| = 2$. Furthermore, they prove that, for finite values of n , the eigenvector $\psi_n^{(N)}$ of $H^{(N)}(x_0)$ tends to the corresponding eigenvector of $H(x_0)$ when $N \rightarrow \infty$.

We now establish the connection between the $H(x_0)$ and the Mathieu differential equation⁵

$$\frac{d^2 y}{dv^2} + (a - 2q \cos(2v))y = 0, \tag{10}$$

where a is the eigenvalue, also known as the characteristic value when the solution y has period of π or 2π . We follow the derivation of Shirts in Ref. 6 and use Floquet's theorem to obtain

$$y = \exp(i\nu v)P(v) = \exp(i\nu v) \sum_k c_{2k} \exp(2ikv), \tag{11}$$

where the Fourier expansion of $P(v)$ has been inserted. This corresponds to the matrix equation for the coefficients c_{2k} compactly written as

$$c_{2k-2} - V_{2k}c_{2k} + c_{2k+2} = 0, \tag{12}$$

where $V_{2k} = [a - (\nu + 2k)^2]/q$. This is identical to the discretized harmonic oscillator Hamiltonian $H(x_0)$ with an eigenvalue λ after identifications

$$\nu = -2x_0, \quad k = j, \quad q = 4/\omega^2, \quad a = 8\lambda/\omega^2, \tag{13}$$

where elements are identified according to $2k \leftrightarrow j$. Thus all results obtained for the discretized harmonic oscillator also hold for Mathieu functions (11) with parameters given in Eq. (13). For $x_0 = 0$ and $x_0 = \pm \frac{1}{2}$ the solutions of $H(x_0)$ can be chosen to be even or odd with respect to j . This corresponds to writing $P(v)$ in terms of sines and cosines. Special attention must be given to the even solutions of $H(x_0 = 0)$, where the resulting equations in the matrix representation read

$$-\psi_1/\sqrt{2} = \lambda_{2n}\psi_0, \tag{14}$$

$$-\psi_0/\sqrt{2} + \omega^2\psi_1/2 - \psi_2/2 = \lambda_{2n}\psi_1, \tag{15}$$

$$-\psi_{j-1}/2 + \omega^2j^2\psi_j/2 - \psi_{j+1}/2 = \lambda_{2n}\psi_j, \quad j \geq 2. \tag{16}$$

The eigenvalues for $x_0 = 0$ correspond to characteristic values $\{a_{2n}(q), b_{2n}(q)\}$, while the case $x_0 = \pm \frac{1}{2}$ is linked to $\{a_{2n+1}(q), b_{2n+1}(q)\}$ as defined in Ref. 5.

The asymptotical expansion of the eigenvalues corresponding to the limit $q \rightarrow \infty$ or $\omega \rightarrow 0$ was obtained by Meixner and Schäfke in Ref. 12. The derivation of the eigenvalues is based on the three-term recurrence relations for the Mathieu functions and the requirement that the norm of the error in the eigenvalue equation vanishes faster than a specific power of ω . Meixner and Schäfke quote the asymptotical characteristic values of the Mathieu equation up to and including the order ω^7 in Theorem 7 in Sec. 2.3. Some error estimates for asymptotical expansions of Mathieu functions by Kurz are given in Ref. 27. Because the Mathieu equation is also the Schrödinger equation of the quantum pendulum or a particle in a periodic potential, it has been studied independently in physics, too.^{10,11,31} Especially, the same general expansion for eigenvalues and several further terms for the ground state energy were obtained by Stone and Reeve in Ref. 11.

In this limit, we can write the eigenvalues of $H(x_0)$ as

$$\lambda_n \sim \sum_{m=0}^{\infty} \lambda_n^{(m)} \omega^m, \tag{17}$$

where $\omega \rightarrow 0$, and

$$\lambda_n^{(m)} = \sum_{k=0}^{m'} \lambda_{n,k}^{(m)} \hat{n}^{m+2(k-m')} \tag{18}$$

with $\hat{n} := 2n + 1$ and $m' = \lfloor m/2 \rfloor$. This structure is identical to that of the Hermite polynomials (5), if one identifies \hat{n} with ξ . By Ref. 12, the eigenvalues (20) do depend on x_0 , but this dependence decreases exponentially as $\omega \rightarrow 0$. The maximal difference is given by¹²

$$\lambda_n(x_0 = \pm \frac{1}{2}) - \lambda_n(x_0 = 0) \sim (-1)^n B_0 (1 - B_1 \omega) \omega^{-n-3/2} \exp(-8/\omega), \tag{19}$$

where B_0 and B_1 depend on n but not on ω .

This allows us to write the eigenvalues of $H(x_0)$ as

$$\lambda_n \sim -1 + \frac{\omega \hat{n}}{2} - \frac{\omega^2 d_2}{2^6} - \frac{\omega^3 d_3}{2^{11}} - \frac{\omega^4 d_4}{2^{17}} - \frac{\omega^5 d_5}{2^{23}} - \frac{\omega^6 d_6}{2^{27}} - \frac{\omega^7 d_7}{2^{33}} - \frac{\omega^8 d_8}{2^{40}} - \frac{\omega^9 d_9}{2^{47}} - \frac{\omega^{10} d_{10}}{2^{51}} \\ - \frac{\omega^{11} d_{11}}{2^{57}} - \frac{\omega^{12} d_{12}}{2^{61}} - \frac{\omega^{13} d_{13}}{2^{69}} - \frac{\omega^{14} d_{14}}{2^{72}} - \frac{\omega^{15} d_{15}}{2^{79}} - \frac{\omega^{16} d_{16}}{2^{87}} + \mathcal{O}(\omega^{17}), \quad (20)$$

where the coefficients d_k read

$$d_2 = \hat{n}^2 + 1,$$

$$d_3 = \hat{n}^3 + 3\hat{n},$$

$$d_4 = 5\hat{n}^4 + 34\hat{n}^2 + 9,$$

$$d_5 = 33\hat{n}^5 + 410\hat{n}^3 + 405\hat{n},$$

$$d_6 = 63\hat{n}^6 + 1260\hat{n}^4 + 2943\hat{n}^2 + 486,$$

$$d_7 = 527\hat{n}^7 + 15617\hat{n}^5 + 69001\hat{n}^3 + 41607\hat{n},$$

$$d_8 = 9387\hat{n}^8 + 388780\hat{n}^6 + 2845898\hat{n}^4 + 4021884\hat{n}^2 + 506979,$$

$$d_9 = 175045\hat{n}^9 + 9702612\hat{n}^7 + 107798166\hat{n}^5 + 288161796\hat{n}^3 + 130610637\hat{n},$$

$$d_{10} = 422565\hat{n}^{10} + 30315780\hat{n}^8 + 480439190\hat{n}^6 + 2135766820\hat{n}^4 + 2249346285\hat{n}^2 + 238353840,$$

$$d_{11} = 4194753\hat{n}^{11} + 379291385\hat{n}^9 + 8186829426\hat{n}^7 + 55529955498\hat{n}^5 + 110241863469\hat{n}^3 \\ + 41540033277\hat{n},$$

$$d_{12} = 10645960\hat{n}^{12} + 1187264199\hat{n}^{10} + 33678377895\hat{n}^8 + 327725946398\hat{n}^6 + 1081358909790\hat{n}^4 \\ + 940077055035\hat{n}^2 + 88258370067,$$

$$d_{13} = 440374207\hat{n}^{13} + 59495737574\hat{n}^{11} + 2155821044201\hat{n}^9 + 28738150160500\hat{n}^7 \\ + 144821249264769\hat{n}^5 + 236410740537606\hat{n}^3 + 78243613727607\hat{n},$$

$$d_{14} = 578183175\hat{n}^{14} + 93209584104\hat{n}^{12} + 4215683624295\hat{n}^{10} + 74269604367684\hat{n}^8 \\ + 537905750769429\hat{n}^6 + 1456767306013752\hat{n}^4 + 1105711550410653\hat{n}^2 \\ + 94839535889532,$$

$$d_{15} = 12308013927\hat{n}^{15} + 2337227706555\hat{n}^{13} + 129437253243675\hat{n}^{11} + 2928506455684095\hat{n}^9 \\ + 29119560960614085\hat{n}^7 + 120372998803922241\hat{n}^5 + 170921920649402745\hat{n}^3 \\ + 51316344023990085\hat{n},$$

$$d_{16} = 530039126159\hat{n}^{16} + 117243302735480\hat{n}^{14} + 7823093961425652\hat{n}^{12} \\ + 222043810819026856\hat{n}^{10} + 2924952921130025194\hat{n}^8 + 17380315268028265224\hat{n}^6 \\ + 40851669411526600980\hat{n}^4 + 27983551470330365784\hat{n}^2 + 2235152520630714879.$$

We obtain the terms for orders $8 \leq m \leq 11$ by exploiting Eq. (18) when explicitly evaluating the determinant of Eq. (7). As a first step, setting $x_0 = 0$ halves the dimension of the tri-diagonal matrix. Next, by translating one of the eigenvalues close to zero by subtracting the known expansion of this eigenvalue, the determinant becomes an essentially linear function of the chosen, translated eigenvalue. The next unknown term is inserted as a parameter and the determinant is calculated for several values of ω , preferably in the form $\{2^{-k}\omega_0\}_{k=0}^{3-5}$. This choice lets us separate the leading correction and the subsequent corrections. In order to obtain the terms d_{2k} and d_{2k+1} , we must correctly determine all eigenvalues λ_n when $n \leq k$. Sufficient accuracy is guaranteed by using the high-precision numerics of MATHEMATICA software. The method for obtaining the orders $m > 11$ requires explicit knowledge on the properties of the eigenvectors and the discussion is postponed until the end of Sec. III.

The asymptotical nature of the expansion means that for each value of ω and n , there exists an optimal order m which minimizes the error in the eigenvalue, i.e., the function

$$\Delta\lambda(\omega, n, m) := \left| \lambda_n - \sum_{m'=0}^m \lambda_n^{(m')} \omega^{m'} \right|, \tag{21}$$

with respect to m . The exact eigenvalue λ_n exists and is finite for all nonzero values of ω according to the Sturmian theory of second-order linear differential equations, see e.g., Ref. 5. In other words, for sufficiently small values of ω the error is dominated by the first omitted term, i.e., $\Delta\lambda(\omega, n, m) \sim |\lambda_n^{(m+1)}| \omega^{m+1}$. Because the asymptotical eigenvalue is divergent, it surely crosses the exact eigenvalue when ω is increased, but this occurs outside the range of asymptotical convergence. Similar asymptotical convergence should be observed for the asymptotical eigenvectors, too. Assuming $\psi_n^{(m, x_0)}$ corresponds to the asymptotical expansion of the eigenvalues up to and including order ω^m , we expect error in the norm to behave as

$$\|\psi_n^{(m, x_0)} - \psi_n^{(x_0)}\| \sim C(n, m) \omega^m, \tag{22}$$

where $\omega \rightarrow 0$ and $C(n, m)$ is a simple function of n and m . Although this has not been proven, Eq. (22) appears to be correct and we will ultimately give an approximate expression for $C(n, m)$, too. Outside the regime of asymptotical convergence the error (22) approaches $\sqrt{2}$ as the asymptotical solution becomes orthogonal to the exact one.

Next we show that the discrete eigenvalue problem Eq. (7) is a meaningful asymptotical limit of the continuous harmonic oscillator equation (1). The problems are identical in the leading infinitesimal order when ω is infinitesimal, but the limit $\omega \rightarrow 0$ is subtle. As long as $\omega > 0$, both the eigenvalues and eigenvectors of the discretized problem tend to those of the continuous harmonic oscillator with this ω . For $\omega = 0$ the continuous problem becomes abruptly the free particle Hamiltonian with solutions

$$\psi_{\omega=0}(x) = e^{ikx}, \quad \lambda_{\omega=0} = k^2/2, \tag{23}$$

where k is the standard name for the wave number. Simultaneously the discretized problem becomes the well-known nearest-neighbor chain with eigenvectors and eigenvalues,

$$\psi_k = \{e^{ik(j-x_0)}\}_j, \quad \lambda_{\omega=0} = -\cos(k). \tag{24}$$

For sufficiently small values of k we have $\lambda_{\omega=0} \approx -1 + k^2/2$, in agreement with Eq. (23). In contrast, we are interested in the bound-state solutions of Eq. (1) and those eigenvectors of the discretized problem that can be uniquely related to these continuous solutions for $\omega > 0$.

The harmonic oscillator is discretized by restricting the values of x onto a countable and evenly spaced subset of \mathbb{R} . The lowest-order central approximation for a second-order derivative is simply

$$\psi''(x) = \frac{\psi(x-h) - 2\psi(x) + \psi(x+h)}{h^2} [+ \mathcal{O}(h^4)]. \tag{25}$$

Assuming $\psi(x)$ to be real analytic allows us to write the numerator of the right-hand side as a Taylor series

$$\psi(x+h) - 2\psi(x) + \psi(x-h) = \sum_{k=1}^{\infty} \frac{2h^{2k}}{(2k)!} \frac{d^{2k}\psi(x)}{dx^{2k}}. \tag{26}$$

If h is infinitesimal and as the derivatives of ψ are finite in all orders, the only remaining term is $h^2\psi''(x)$. Thus, in the lowest infinitesimal order the discretized eigenvalue problem gives a second-order differential equation

$$-\frac{1}{2} \frac{d^2\psi_x}{dx^2} + \frac{\omega^2 x^2}{2h^2} \psi_x = h^{-2}(-1 + \lambda) \psi_x \tag{27}$$

which is identical to Eq. (1) apart from the constant $-h^{-2}$ and the redefinitions $\omega \mapsto \omega/h$ and $\lambda \mapsto \lambda/h^2$. The discreteness of the problem can also be varied by rescaling the value of ω . Thus, instead of decreasing the size h of the steps, we set $h=1$ and let $\omega \rightarrow 0$. From Eq. (27) we see that asymptotically the eigenvalues and eigenvectors have the form $\lambda_n \sim -1 + \omega(n+1/2)$ and $\psi_x \sim \psi(x)$, as expected.

We have already pointed out that the matrix $H(x_0)$ in Eq. (7) can be derived from the Mathieu equation. The underlying reason for this is that the problems are canonically conjugate. Inserting the full expansion Eq. (26) into Eq. (27) yields an obvious differential equation in ψ_x with respect to x . The canonical transformation $id/dx \rightarrow \bar{v}$ and $x \rightarrow -id/d\bar{v}$ preserves the eigenvalues and produces the differential equation

$$-\frac{\omega^2}{2} \frac{d^2\psi_{\bar{v}}}{d\bar{v}^2} - \left(\sum_{k=0}^{\infty} \frac{(-1)^k \bar{v}^{2k}}{(2k)!} \right) \psi_{\bar{v}} = \lambda \psi_{\bar{v}}. \tag{28}$$

Noticing that the sum is equal to $\cos(\bar{v})$ and setting $v := (\bar{v} + \pi)/2$, we obtain the canonical form of the Mathieu equation with parameters given in Eq. (13).

After these important preliminaries, we are able to proceed towards the actual solution for the discretized harmonic oscillator. In order to treat eigenvectors of all matrices $H(x_0)$ on an equal footing, we replace the index j by $x := j - x_0$. For arbitrary values of x_0 and j the new index x becomes a continuous one on \mathbb{R} . We thus obtain functions $\psi_x^{(n)}$, where n is the state index. We propose that these functions $\psi_x^{(n)}$ are real-analytic and that they give the eigenvectors of $H(x_0)$ asymptotically, i.e.,

$$\psi_n^{(x_0)} \sim \{ \psi_{j-x_0}^{(n)} \}_{j=-\infty}^{\infty} \tag{29}$$

when $\omega \rightarrow 0$. The problem tends to the continuous one in the lowest (infinitesimal) approximation in ω . Thus it is reasonable to assume that the lowest-order approximation for the solution functions is given by $\psi_x^{(n)} \sim \psi_n(x)$ as $\omega \rightarrow 0$.

The general form of the asymptotical solution of the discretized harmonic oscillator now reads

$$\psi_x^{(n)} \propto \exp \left(\sum_{k=1}^{\infty} \sum_{l=k}^{\infty} \alpha_{kl}^{(n)} \omega^{l-1} \xi^{2k} \right) \sum_{k=0}^{k'} \sum_{l=1}^{\infty} (h_k^{(n)} \omega^{l-1} \beta_{kl}^{(n)} \xi^{n+2(k-k')}), \tag{30}$$

where $\alpha_{kl}^{(n)}$ and $\beta_{kl}^{(n)}$ are constants to be determined. The solution to the continuous case yields $\alpha_{1,1}^{(n)} = -1/2$ and $\beta_{k,1}^{(n)} = 1$. We are free to normalize the solution so we can choose $\beta_{0,l}^{(n)} = 0$ for $l > 1$.

The main point of introducing the functions $\psi_x^{(n)}$ is that they transform the difference-equation-type eigenvalue problem corresponding to the discretized harmonic oscillator into an infinite set of algebraic equations for each value of n . The eigenvalues (20) appear as parameters and they are required in order to solve the equations for the sets of coefficients $\{\alpha_{kl}^{(n)}\}$ and $\{\beta_{kl}^{(n)}\}$. Fortunately, the equations uniquely determine every single coefficient. Because the expansion of the eigenvalues is asymptotical, the meaning of the full solution to these equations must be determined later.

In practice, we need a suitable truncation of Eq. (30) and thus we define an (un-normalized) approximate eigenvector

$$\psi_n^{(m,x_0)} := \{\psi_{j-x_0}^{(n,m)}\}_{j=-\infty}^{\infty}, \tag{31}$$

where $\psi_x^{(n,m)}$ contains only those terms with $l \leq m$. The definition of $\psi_n^{(1,x_0)}$ obviously coincides with the continuous solution at x_0 . In numerical calculations, and always for even values of m , we must truncate the eigenvector with respect to j , by setting $(\psi_n^{(m,x_0)})_j = 0$ for components $|j| > j_0$ with a sufficiently large value of j_0 .

We now give the infinite set of algebraic equations corresponding to the transformation of the difference equation when the solution functions $\psi_x^{(n)}$ are substituted into the eigenvalue equation. Rearranging the terms, we find that each equation can be written in the form

$$\frac{\psi_{x-1}^{(n)} + \psi_{x+1}^{(n)}}{2} = \psi_x^{(n)}(-\lambda_n + \omega^2 x^2/2), \tag{32}$$

where $x = j - x_0$. Inserting the general ansatz (30) into Eq. (32) expresses the equation in terms of ξ and ω . The exponential part of the ansatz on the right-hand side canceled simply by subtracting the corresponding exponent from those on the left-hand side. This yields an equation

$$\begin{aligned} & \frac{1}{2} \left[\exp\left(\sum_{k=1}^{\infty} \sum_{l=k}^{\infty} \alpha_{kl}^{(n)} \omega^{k+l-1} [(x-1)^{2k} - x^{2k}]\right) \sum_{k=0}^{k'} \sum_{l=1}^{\infty} (h_k^{(n)} \omega^{l-1} \beta_{kl}^{(n)} [\sqrt{\omega}(x-1)]^{n+2(k-k')}) \right. \\ & \quad \left. + \exp\left(\sum_{k=1}^{\infty} \sum_{l=k}^{\infty} \alpha_{kl}^{(n)} \omega^{k+l-1} [(x+1)^{2k} - x^{2k}]\right) \sum_{k=0}^{k'} \sum_{l=1}^{\infty} (h_k^{(n)} \omega^{l-1} \beta_{kl}^{(n)} [\sqrt{\omega}(x+1)]^{n+2(k-k')}) \right] \\ & = \left[-\sum_{m=0}^{\infty} \lambda_{n(m)} \omega^m + \frac{\omega^2 x^2}{2} \right] \sum_{k=0}^{k'} \sum_{l=1}^{\infty} (h_k^{(n)} \omega^{l-1} \beta_{kl}^{(n)} [\sqrt{\omega} x]^{n+2(k-k')}). \end{aligned} \tag{33}$$

These equations are then expanded as functions of x and ω as the resulting equations are easier to solve. The equations must hold for all values of the linearly independent variables x and ω so each equation must be solved separately. For the purposes of generality, it would be preferable to expand with respect to ξ , but the resulting equations are much more difficult, both to obtain and to solve. Nevertheless, the obtained solution can be inserted into to these equations in order to show that the results are correct. This will be done in Sec. V.

In order to obtain the eigenvector $\psi_n^{(m,x_0)}$ we must solve and satisfy all equations corresponding to

$$\{\{\omega^{m'} \xi^{n+2m'-2l'}\}_{l'=0}^{m'+k'}\}_{m'=0}^m. \tag{34}$$

This is of course done recursively, by inserting the known part of the solution and solving for the next level. In order to connect Eq. (34) with the order of the solution, we state that the equations corresponding to a fixed value of m' uniquely determines the coefficients with $l = m'$.

After a while, one starts to see regularities in the coefficients and attempts to express these in a functional form. We have been able to find rather general expressions for the coefficients. This

means that the coefficients have been expressed in terms of n and the order of the expansion, whenever this is possible. We have conjectured the general form of the terms which means that we know how far away we are from obtaining further terms.

We have found that the functions $\psi_x^{(n,m)}(\xi)$ are asymptotical solutions of the differential equation

$$\left(-\sum_{k=0}^m \frac{\omega^k}{(2k)!} \frac{d^{2k}}{d\xi^{2k}} + \frac{\omega \xi^2}{2} \right) \psi_x^{(n,m)} = \sum_{k=0}^m \lambda_n^{(k)} \omega^k \psi_x^{(n,m)} \tag{35}$$

in a specific sense. After all derivatives have been carried out, the terms multiplying the common exponential factor cancel up to and including the order ω^m . If the solution $\psi_x^{(n,m-1)}$ is known, we obtain an explicit differential equation for the exponential part $\omega^{m-1} f_m(\xi)$, the correction to the Hermite polynomial $\omega^{m-1} g_m(\xi)$ and the energy eigenvalue $\lambda_n^{(m)}$. In case of the ground state and the first excited state ($n=1$), the condition, $g_m(\xi)=0$ for $m > 1$, renders the problem solvable. For $n \geq 2$ we must insert the ansatz (30) in order to obtain the solution.

The results are, naturally, in complete agreement with those obtained by using the difference equation. We are using just another representation of the original problem. Equation (35) enables us to obtain the solutions for fixed values of n up to relatively high orders with respect to powers of ω . Thus we can both extend the general expression for the eigenenergies in Eq. (20) and those for the coefficients in the exponential part of the solutions. For the ground state energy, we find the terms beyond order ω^{16} to be

$$\begin{aligned} & - \frac{363372562420411197 \omega^{17}}{2^{79}} - \frac{6258692522467212813 \omega^{18}}{2^{83}} - \frac{227867608383920243815 \omega^{19}}{2^{88}} \\ & - \frac{4372199488222446620121 \omega^{20}}{2^{92}} - \frac{352807992522448740907163 \omega^{21}}{2^{98}} \\ & - \frac{7465886451386334274097895 \omega^{22}}{2^{102}} - \frac{330752735437897260202410959 \omega^{23}}{2^{107}} \\ & - \frac{7654237307570898665851927581 \omega^{24}}{2^{111}} - \frac{1477812451863756884805687589129 \omega^{25}}{2^{118}} \\ & - \frac{37132718819258763418452357390369 \omega^{26}}{2^{122}} \\ & - \frac{1939848955425261040700592191917783 \omega^{27}}{2^{128}} \\ & - \frac{52598573101029275526869814635336865 \omega^{28}}{2^{131}} \\ & - \frac{5914101566562517015636997146651378649 \omega^{29}}{2^{137}} \\ & - \frac{172129355454985486683952198830698506149 \omega^{30}}{2^{141}} \\ & - \frac{10362392343003738344189045786484697182753 \omega^{31}}{2^{146}} + \mathcal{O}(\omega^{32}). \tag{36} \end{aligned}$$

The corresponding asymptotical eigenvector contains $31(1+31)/2=496$ linearly independent terms. The coefficient of $\omega^{30} \xi^2$ in the exponential part reads

$$-\frac{5207328980459439428858189871778019425519567564728193}{2765292404617797269550429065808396826741571584}. \tag{37}$$

The general solution is given in the next section.

III. THE GENERAL SOLUTION OF THE DISCRETIZED HARMONIC OSCILLATOR

For reasons of completeness and easy accessibility some of the definitions will be repeated in this section. The m th order solution function $\psi_x^{(n,m)}$, corresponds all terms up to and including $l=m$ in Eq. (30). The asymptotical expansion of the eigenvalues λ_n is given in Eq. (20). The state index n determines two expansion parameters,

$$\hat{n} := 2n + 1 \quad \text{and} \quad k' := \lfloor n/2 \rfloor, \tag{38}$$

where $k' = n/2$ if n is even and $k' = (n - 1)/2$ if n is odd. The gamma function $\Gamma(x)$ is the generalized factorial with the defining property $x\Gamma(x) = \Gamma(x + 1)$. We need the values for integer and half-integer values which read

$$\Gamma(k) = (k - 1)!, \quad \Gamma(k + \frac{1}{2}) = 2^{-k} \sqrt{\pi} (2k - 1)!!, \tag{39}$$

where the double factorial $k!!$ for integer values of k is given by $k(k - 2) \times \dots \times (1 \text{ or } 2)$. The coefficients in the Hermite polynomials simplify to

$$h_k^{(n)} = \frac{(-1)^{k'+k} 2^{2k+(1-(-1)^n)/2} n!}{(2k+(1-(-1)^n)/2)!(k'-k)!}. \tag{40}$$

A convenient normalization for the eigenvectors is obtained by requiring that

$$\psi_x^{(n)} \sim \xi^{(1-(-1)^n)/2}, \quad x \rightarrow 0, \tag{41}$$

i.e., ~ 1 for even values of n and $\sim \xi$ for odd values of n . Also bear in mind that

$$\alpha_{1,1}^{(n)} = -1/2, \quad \beta_{k,1}^{(n)} = 1, \quad \text{and} \quad \beta_{0,l(>1)}^{(n)} = 0. \tag{42}$$

Under these constraints we have conjectured that the general form of the coefficients. In the exponential part,

$$\exp\left(\sum_{k=1}^{\infty} \sum_{l=k}^{\infty} \alpha_{kl}^{(n)} \omega^{l-1} \xi^{2k}\right), \tag{43}$$

the coefficients can be written as

$$\alpha_{k,k+l}^{(n)} = \sum_{l'=0}^l \alpha_{k,k+l}^{[l']1} \hat{n}^{l'}. \tag{44}$$

Please note that if the coefficient $\alpha_{kl}^{(n)}$ are written as polynomials in n instead of \hat{n} , the signs of the corresponding expansion coefficients $\tilde{\alpha}_{k,k+l}^{[l']1}$ appear to be given by $(-1)^k$. An efficient way to write these coefficients is given by

$$\alpha_{k,k+l}^{(n)} = (-1)^k 2^{-2k} \left(\sum_{l'=0}^{\lfloor l/2 \rfloor} \frac{\Gamma(k+1/2) \mathcal{Q}(k,l,2k-5+l') \hat{n}^{l-2l'}}{\Gamma(k+l) \sqrt{\pi}} + \sum_{l'=0}^{\lfloor (l-1)/2 \rfloor} \bar{\mathcal{Q}}(k,l,k-2+l') \hat{n}^{l-1-2l'} \right), \tag{45}$$

where $Q(k, l, 2k - 5 + l')$ and $\bar{Q}(k, l, k - 2 + l')$ are polynomials in k of orders $2k - 5 + l'$ and $k - 2 + l'$, respectively. An important consequence of Eq. (45) is that regardless of the values of l and n we have

$$\lim_{k \rightarrow \infty} \alpha_{k+1, k+1+l}^{(n)} / \alpha_{k, k+l}^{(n)} = -1/4. \tag{46}$$

The explicit expressions for the seven leading coefficients have been obtained and they read

$$\alpha_{kk}^{(n)} = \frac{(-1)^k 2^{2-2k} \Gamma(k+1/2)}{k(2k-1)^2 \Gamma(k) \sqrt{\pi}},$$

$$\alpha_{k, k+1}^{(n)} = (-1)^k 2^{-2-2k} \left(\frac{1}{k} + \frac{\Gamma(k+1/2)}{\Gamma(k+1) \sqrt{\pi}} \frac{\hat{n}}{k} \right),$$

$$\alpha_{k, k+2}^{(n)} = (-1)^k 2^{-4-2k} \left(\hat{n} + \frac{\Gamma(k+1/2)}{24\Gamma(k+2) \sqrt{\pi}} [(3+52k+40k^2) + (9+12k)\hat{n}^2] \right),$$

$$\alpha_{k, k+3}^{(n)} = (-1)^k 2^{-9-2k} \left\{ (-1+7k+5k^2) + (3+4k)\hat{n}^2 + \frac{\Gamma(k+1/2)}{24\Gamma(k+3) \sqrt{\pi}} \times [(243+1119k + 1928k^2 + 1376k^3 + 320k^4)\hat{n} + (33+101k+104k^2+32k^3)\hat{n}^3] \right\},$$

$$\alpha_{k, k+4}^{(n)} = (-1)^k 2^{-14-2k} \left\{ (53+120k+136k^2+40k^3)\hat{n} + (37+72k+32k^2)\hat{n}^3/3 + \frac{\Gamma(k+1/2)}{48\Gamma(k+4) \sqrt{\pi}} \times [(-2612925-5292132k+10675063k^2+36766856k^3+40148416k^4+21300608k^5 + 5544448k^6+565760k^7)/315 + (11070+60044k+130810k^2+142112k^3+81280k^4 + 23168k^5+2560k^6)\hat{n}^2 + (585+2288k+3585k^2+2696k^3+960k^4+128k^5)\hat{n}^4] \right\},$$

$$\alpha_{k, k+5}^{(n)} = (-1)^k 2^{-20-2k} \left\{ (-5187-672k+6580k^2+7684k^3+3164k^4+452k^5)/3 + (1214+3744k + 4080k^2+1968k^3+320k^4)\hat{n}^2 + (345+808k+576k^2+128k^3)\hat{n}^4/3 + \frac{\Gamma(k+1/2)}{48\Gamma(k+5) \sqrt{\pi}} \times [(740893230+3944788389k+9627147810k^2+14943869467k^3+15287941200k^4 + 10116675072k^5+4238798592k^6+1079918592k^7+152076288k^8+9052160k^9)\hat{n}/315 + (1825740+11037114k+27955236k^2+37919062k^3+30169312k^4+14491648k^5 + 4122880k^6+636928k^7+40960k^8)\hat{n}^3/3 + (85050+381087k+729798k^2+752369k^3 + 447024k^4+152576k^5+27648k^6+2048k^7)\hat{n}^5/5] \right\},$$

$$\alpha_{k,k+6}^{(n)} = (-1)^k 2^{-26-2k} \left\{ (378033 + 496368k + 786528k^2 + 710816k^3 + 339904k^4 + 79552k^5 + 7232k^6)\hat{n}/3 + (69714 + 241312k + 303392k^2 + 177696k^3 + 49408k^4 + 5120k^5)\hat{n}^3/3 + (17217 + 45360k + 40960k^2 + 15360k^3 + 2048k^4)\hat{n}^5/15 + \frac{\Gamma(k+1/2)}{180\Gamma(k+6)\sqrt{\pi}} \times [(-24640192386810 - 105728184475128k - 155775948330744k^2 - 74654535511116k^3 + 74660144680858k^4 + 156803802177352k^5 + 134434233033760k^6 + 70722102090816k^7 + 24590691451392k^8 + 5680345583616k^9 + 839668527104k^{10} + 71921254400k^{11} + 2714009600k^{12})/9009 + (22093103970 + 162201234402k + 504160865145k^2 + 882850470198k^3 + 986932878421k^4 + 745434338828k^5 + 388089936864k^6 + 138972684672k^7 + 33504543744k^8 + 5179637760k^9 + 462565376k^{10} + 18104320k^{11})\hat{n}^2/21 + (152041050 + 991922940k + 2784482730k^2 + 4353707520k^3 + 4203836660k^4 + 2632731680k^5 + 1088777440k^6 + 294912320k^7 + 50245120k^8 + 4874240k^9 + 204800k^{10})\hat{n}^4 + (2606310 + 12799746k + 27798345k^2 + 34245070k^3 + 26181505k^4 + 12857468k^5 + 4055200k^6 + 792320k^7 + 87040k^8 + 4096k^9)\hat{n}^6] \right\}.$$

Thus, for an arbitrary order m , we can obtain the expressions for coefficients corresponding to $\{\omega^{m-1} \xi^{2(m-l')}\}_{l'=0}^6$. Furthermore we find

$$\alpha_{1,8}^{(n)} = -(505549159\hat{n} + 177209155\hat{n}^3 + 8289645\hat{n}^5 + 40329\hat{n}^7)/2^{37} - (-2741702 + 12248825\hat{n}^2 + 1518052\hat{n}^4 + 26073\hat{n}^6)/2^{32},$$

$$\alpha_{1,9}^{(n)} = -(-840819020949 + 1419128841068\hat{n}^2 + 221074444682\hat{n}^4 + 6195597884\hat{n}^6 + 21259875\hat{n}^8)/(3 \times 2^{47}) - (1318785849\hat{n} + 459389255\hat{n}^3 + 29718111\hat{n}^5 + 335617\hat{n}^7)/2^{38},$$

$$\alpha_{2,9}^{(n)} = (131257276187\hat{n} + 37843099187\hat{n}^3 + 1323046497\hat{n}^5 + 4456305\hat{n}^7)/2^{44} + (48228434 + 93959845\hat{n}^2 + 8787700\hat{n}^4 + 110661\hat{n}^6)/2^{34}.$$

The exponential part (43) is now completely determined up to the ninth order, i.e., known for arbitrary values of n for terms with $l \leq 9$.

If we exclude the dependence on 2^{-2k} and also that given by the gamma functions in the coefficients, we observe a very distinct regularity. The dependence of the leading power k in each polynomial sequence starting from \hat{n}^0 for a fixed value l in $\alpha_{k,k+l}^{(n)}$ and going upwards by one both for l and the power of \hat{n} is so far always given by

$$\{\tilde{\alpha}_{(l,l')}\}_{l'} = \{\tilde{\alpha}_{(l,0)}/(4^{l'}(l')!)\}. \tag{47}$$

The initial values in cases $l' \leq 5$ are given by

$$\{\tilde{\alpha}_{(l,0)}\}_{l=0}^5 = \{1, 1/4, 5/48, 5/512, 221/96768, 113/786432\}. \tag{48}$$

This dependence is by no means proven but it corroborates our choice for the prefactors in Eq. (45). A bit surprisingly, we find that the coefficients $\alpha_{k,k+l}^{(n)}$ in Eq. (44) contain information about the energy eigenvalues in cases $n=0$ and $n=1$, i.e.,

$$\alpha_{-1,m}^{(n=0 \text{ or } 1)} = -\lambda_{n=0 \text{ or } 1}^{(m)} \tag{49}$$

The right-hand side is taken from Eq. (20) and it is correctly reproduced for $m \leq 6$.

The coefficients $\{\beta_{kl}^{(n)}\}$ determine a set of new polynomials, where the coefficients multiplying the powers of ξ depend on ω . In the limit $\omega \rightarrow 0$ these polynomials tend to the Hermite polynomials. They are, unquestionably, a new class of generalized Hermite polynomials. They are defined as parts of the eigenvectors of a Hermitian matrix. Because the exponential part is rather complicated, the measure, with respect to which they become asymptotically orthogonal, is necessarily a complicated one. It depends on both of the eigenvectors, i.e., it is not a measure in the classical sense at all. No simple recursion relation for the polynomials is yet known, and we do not know, whether they satisfy any differential equation of finite order. This means that they could be an example of the second category of generalized Hermite polynomials as defined by Borzov in Ref. 20. Such discussion is beyond the scope of the present study and we will concentrate on simpler properties of the polynomials.

Our generalized Hermite polynomials are defined as

$$H_n^\omega(\xi) := \sum_{k=0}^{k'} \tilde{h}_k^{(n)} \xi^{n+2(k-k')}, \tag{50}$$

where the modified coefficients are given by

$$\tilde{h}_k^{(n)} := h_k^{(n)} \sum_{l=1}^{\infty} (\omega^{l-1} \beta_{kl}^{(n)}). \tag{51}$$

Because the generalized Hermite polynomials $H_n^\omega(\xi)$ fix the nodes (zeroes) of the functions $\psi_x^{(n)}$, it is equally important to obtain correct polynomials as it is to obtain the correct exponential factors.

We conjecture that the general form of the coefficients $\{\beta_{kl}^{(n)}\}$ reads

$$\beta_{kl}^{(n)} = \sum_{l'=0}^{l-1} \left(\sum_{\bar{l}=1}^{2(l-1)-l'} (\rho_{l',\bar{l}}^{(l)} + [(1-(-1)^n]/2] \bar{\rho}_{l',\bar{l}}^{(l)}) k^{\bar{l}} \right) (k')^{l'}, \tag{52}$$

where $\rho_{l',\bar{l}}^{(l)}$ and $\bar{\rho}_{l',\bar{l}}^{(l)}$ are constants. Additionally, $\bar{\rho}_{l',\bar{l}}^{(l)} = 0$ when $\bar{l} = 2(l-1) - l'$ or $l' = l-1$. This expansion with respect to k and k' shows that even and odd values of n should be treated separately.

Some general properties of the coefficients $\rho_{l',\bar{l}}^{(l)}$ and $\bar{\rho}_{l',\bar{l}}^{(l)}$ have been gleaned. The recurring appearance of the factor $(10k' - k)$ is by far the most striking of the observed regularities. This factor may, in time, explain some properties generalized Hermite polynomials. We conjecture that

$$\sum_{l'=0}^{l-1} (\rho_{l',2(l-1)-m_0-l'}^{(l)} k^{2(l-1)-m_0-l'} n^{l'}) = k^{l-1-m_0} (10k' - k)^{l-1-2m_0} P(2m_0, l), \tag{53}$$

where $2m_0 < l$ and $P(2m_0, l)$ denotes a $(2m_0)$ th order polynomial in k and k' . Similarly, the difference between even and odd values of n corresponds to

$$\sum_{l'=0}^{l-2} (\bar{\rho}_{l',2l-3-m_0-l'}^{(l)} k^{2l-3-m_0-l'} n^{l'}) = k^{l-1-m_0} (10k' - k)^{l-2-2m_0} \bar{P}(2m_0, l), \tag{54}$$

where $2m_0 < l - 1$ and $\bar{P}(2m_0, l)$ again denotes a $(2m_0)$ th order polynomial in k and k' .

In the leading and next-to-leading orders the polynomials $P(0, l)$, $P(2, l)$, $\bar{P}(0, l)$, and $\bar{P}(2, l)$ have been explicitly evaluated. Thus, we define the quantities

$$B(l, 2l - 2) := \frac{k^{l-1}(10k' - k)^{l-1}}{48^{l-1}(l-1)!} \tag{55}$$

and

$$B(l, 2l - 3) := \frac{k^{l-2}(10k' - k)^{l-3}}{5 \times 48^{l-1}(l-2)!} [(l-2)658(k')^2 + (402 - 126l)k'k + (8l - 31)k^2], \tag{56}$$

which has been confirmed up to the sixth order, i.e., $l = 6$. Similarly, the leading differences give rise to the quantities

$$\bar{B}(l, 2l - 3) := \frac{4k^{l-1}(10k' - k)^{l-2}}{48^{l-2}(l-2)!} \tag{57}$$

and

$$\bar{B}(l, 2l - 4) := \frac{k^{l-2}(10k' - k)^{l-4}}{5 \times 48^{l-1}(l-3)!} [(2632l - 2576)(k')^2 + (1470 - 504l)k'k + (32l - 145)k^2]. \tag{58}$$

The leading terms are very similar, but also the next-to-leading terms $B(l, 2l - 3)$ and $\bar{B}(l, 2l - 4)$ share several common features. Most importantly, the l -dependence in the polynomial section is identical, apart from a factor of 4.

Below, we give the explicit values of the coefficients $\beta_{kl}^{(n)}$ in cases $2 \leq l \leq 7$. It is convenient to separate the even and odd values of n , because the correct expansion parameter appears to be k' . Please note that these expressions automatically yield $\beta_{0,l}^{(n)} = 0$ for $l > 1$,

$$\beta_{k,2}^{(2k')} = [(3k - k^2) + (10k)k'] / 48,$$

$$\beta_{k,2}^{(2k'+1)} = \beta_{k,2}^{(2k')} + k/12,$$

$$\beta_{k,3}^{(2k')} = [(855k - 64k^2 - 14k^3 + 5k^4) + (784k + 48k^2 - 100k^3)k' + (1316k + 500k^2)(k')^2] / 23040,$$

$$\beta_{k,3}^{(2k'+1)} = \beta_{k,3}^{(2k')} + [(249k + 49k^2 - 20k^3) + (532k + 200k^2)k'] / 11520,$$

$$\beta_{k,4}^{(2k')} = B(4, 6) + B(4, 5) + [(371385k - 203498k^2 - 12129k^3 + 1438k^4) + (1110698k + 102042k^2 - 26252k^3)k' + (496932k + 93984k^2)(k')^2 + 560200k(k')^3] / 23224320,$$

$$\beta_{k,4}^{(2k'+1)} = \beta_{k,4}^{(2k')} + \bar{B}(4, 5) + \bar{B}(4, 4) + [(67680k + 12347k^2 - 2602k^3) + (108544k + 33762k^2)k' + 114456k(k')^2] / 3870720,$$

$$\beta_{k,5}^{(2k')} = B(5, 8) + B(5, 7) + [(278751375k - 202014918k^2 + 35222268k^3 + 4026748k^4 + 28158k^5 - 9944k^6) + (713250468k - 281790420k^2 - 61452368k^3 - 196176k^4 + 209856k^5)k' + (1105743252k + 198178852k^2 - 10630680k^3 - 275344k^4)(k')^2 + (319197168k + 81282336k^2 - 22799744k^3)(k')^3 + (271672512k + 148408976k^2)(k')^4] / 22295347200,$$

$$\begin{aligned} \beta_{k,5}^{(2k'+1)} = & \beta_{k,5}^{(2k')} + \bar{B}(5,7) + \bar{B}(5,6) + [(119817225k - 23468037k^2 - 12060122k^3 - 330312k^4 \\ & + 100198k^5) + (436319556k + 103769756k^2 - 5886336k^3 - 1921112k^4)k' \\ & + (321608148k + 118383396k^2 - 904080k^3)(k')^2 + (224147856k + 120486304k^2) \\ & \times (k')^3] / 11147673600, \end{aligned}$$

$$\begin{aligned} \beta_{k,6}^{(2k')} = & B(6,10) + B(6,9) + [(134035780725k - 166751340588k^2 + 39327194883k^3 \\ & - 2269605874k^4 - 477614210k^5 - 19226552k^6 + 221782k^7 + 484k^8) + (413990823078k \\ & - 217584747090k^2 + 22678956764k^3 + 8841166604k^4 + 479019924k^5 - 6549884k^6 \\ & + 566984k^7)k' + (526339688532k - 155591533528k^2 - 55003198072k^3 - 3518713436k^4 \\ & + 85514000k^5 - 24903296k^6)(k')^2 + (556945898088k + 131085561976k^2 \\ & + 2790556248k^3 - 280060176k^4 + 424040144k^5)(k')^3 + (116760015552k \\ & + 34523271136k^2 - 5865150192k^3 - 3338174576k^4)(k')^4 + (79966766400k \\ & + 46102886720k^2 + 10162787360k^3)(k')^5] / 11771943321600, \end{aligned}$$

$$\begin{aligned} \beta_{k,6}^{(2k'+1)} = & \beta_{k,6}^{(2k')} + \bar{B}(6,9) + \bar{B}(6,8) + [(34460588160k - 26910050283k^2 + 72069996k^3 \\ & + 1282383895k^4 + 103465570k^5 - 948002k^6 - 316976k^7) + (216801198648k \\ & - 21671791146k^2 - 18471533106k^3 - 1699322576k^4 + 78651584k^5 + 5865684k^6)k' \\ & + (345295895928k + 96181762100k^2 + 1478206984k^3 - 1409258180k^4 + 60822872k^5) \\ & \times (k')^2 + (160052617776k + 64547633160k^2 + 3204992824k^3 - 1898232512k^4)(k')^3 \\ & + (83156900448k + 47130830560k^2 + 10276562912k^3)(k')^4] / 5885971660800, \end{aligned}$$

$$\begin{aligned} \beta_{k,7}^{(2k')} = & B(7,12) + B(7,11) + [(21167446950775125k - 34318046368345140k^2 \\ & + 13674300462898392k^3 - 1352901404372446k^4 - 2843855572731k^5 \\ & + 11311875159790k^6 + 704407032828k^7 + 12949326156k^8 - 177366189k^9 \\ & + 37677640k^{10}) + (59570630372492640k - 60644270495554704k^2 \\ & + 10066261151648252k^3 + 104602336760652k^4 - 246415137367020k^5 \\ & - 20207362771548k^6 - 460168946016k^7 + 2604105504k^8 - 2483040560k^9)k' \\ & + (99669485611466412k - 39020273844707836k^2 + 2200698814542984k^3 \\ & + 2070713072954600 + 212428368788100k^5 + 5622413614220k^6 + 82814211480k^7 \\ & + 70784553840k^8)(k')^2 + (82488078028378080k - 18442822328400480k^2 \\ & - 9100818756007520k^3 - 964844434165920k^4 - 24156442527360k^5 - 3013682511840k^6 \\ & - 1116228072960k^7)(k')^3 + (66588038149135200k + 18345507366303440k^2 \\ & + 1191268975557840k^3 + 10518809509520k^4 + 42474114642960k^5 \\ & + 10244315921840k^6)(k')^4 + (10839030004200960k + 3516288982521792k^2 \\ & - 363895953410496k^3 - 304221200739456k^4 - 51610667908800k^5)(k')^5 \end{aligned}$$

$$+ (6218212960526208k + 3705496740373376k^2 + 898601964676416k^3 + 110684037464000k^4)(k')^6]/1542595452862464000,$$

$$\begin{aligned} \beta_{k,7}^{(2k'+1)} = & \beta_{k,7}^{(2k')} + \bar{B}(7,11) + \bar{B}(7,10) + [(-460686821541975k - 1941941074537755k^2 \\ & + 366877584331212k^3 + 41494582964306k^4 - 9965970910165k^5 - 1112954021925k^6 \\ & - 34972438722k^7 + 958101144k^8 + 3798795k^9) + (9145976126266080k \\ & - 3823250702059872k^2 - 191143081971676k^3 + 173415983605096k^4 \\ & + 24948282593576k^5 + 815890000796k^6 - 41300603344k^7 + 1974092120k^8)k' \\ & + (19045045703842332k - 607795420829248k^2 - 1422456568355676k^3 \\ & - 195873935369616k^4 - 2655255816252k^5 + 611948653316k^6 - 98001423520k^7) \\ & \times (k')^2 + (18941405236672032k + 5863460843089248k^2 + 345511721120640k^3 \\ & - 55753844615456k^4 - 654723253184k^5 + 1792716525600k^6)(k')^3 \\ & + (6296099301099168k + 2692631923242160k^2 + 232744611197904k^3 \\ & - 59049744898144k^4 - 14606140268720k^5)(k')^4 + (2605202959125888k \\ & + 1525593370591680k^2 + 365590616623232k^3 + 44670372947200k^4) \\ & \times (k')^5]/257099242143744000. \end{aligned}$$

In combination with the exponential parts these coefficients determine explicit, analytical expressions for solution function $\psi_x^{(n,7)}$ for arbitrary values of n .³²

It must be re-emphasized that Eq. (30) is an asymptotical solution. Two partially overlapping reasons for this behavior must be stated. First, the solution depends on two independent length scales, i.e., x and ω , and second, the coordinate transformation $x \mapsto \xi = \sqrt{\omega}x$ is singular at $\omega = 0$. These points are rather extensively covered in Ref. 13. The eigenvalues are asymptotically exact for even values of n at $x_0 = 0$ and, probably, for odd values of n at $x_0 = \pm \frac{1}{2}$. Because the error decays exponentially in $1/\omega$, this dependence on x_0 vanishes much before the asymptotical behavior of Eq. (22), i.e., $\|\psi_n^{(m,x_0)} - \psi_n^{(x_0)}\| \sim C(n,m)\omega^m$, appears.

Comparison against numerically obtained eigenstates allows us to give an approximate expression for the function $C(n,m)$. The validity of the calculations is limited by the numerical precision, i.e., to norms of the order of $10^{-11} - 10^{-12}$. We have employed the reliable diagonalization routines of MATLAB software for this purpose. We have studied eigenvectors up to $n \approx 40 - 50$ and the corresponding asymptotical solutions $\psi_n^{(m,x_0)}$ up to the fifth order. A reasonable, order-of-magnitude estimate for the error in the Euclidean norm, when $n \leq 40$, is given by

$$C(n,m) \approx c_m \hat{n}^{2m}, \tag{59}$$

where

$$c_1 = 0.03, \quad c_2 = 0.002, \quad c_3 = 0.0006, \quad c_4 = 1.5 \times 10^{-6}, \quad \text{and} \quad c_5 = 3 \times 10^{-8}. \tag{60}$$

There is a slight difference between even and odd cases, but this is insignificant in an estimate like this. The value of c_5 is set to fit the observed trend in the other coefficients as the asymptotical behavior is only glimpsed. In cases $m = 2$ and $m = 4$, it is vitally important to remember to truncate the asymptotical eigenvector $\psi_n^{(m,x_0)}$ correctly.

For larger values of n , one needs very small values of ω in order to obtain accurate or even reasonable results. But for relatively small values of n , say $n \leq 10$, the error is extremely small at

$\omega \approx 0.01$. The strong dependence on n means that the first few states can be obtained to a high precision even for quite strong couplings in the neighborhood of $\omega \approx 0.1$. We have determined the ground state $n=0$ up to the 31st order and numerical comparison strongly supports the asymptotical behavior ω^m for $m \leq 13$.

In order to make the above discussion more concrete, we explicitly give the second-order solutions as functions of ω , $\xi = \sqrt{\omega} x$, n (not \hat{n}) and k' . For even values of n we find the solution function

$$\psi_x^{(n,2)} = A_{n,x_0} \exp\left(-\left(\frac{1}{2} + (3+2n)\omega/32\right)\xi^2 + (\omega/96)\xi^4\right) \sum_{k=0}^{k'} (h_k^{(n)} \xi^{2k} (1 + (3k - k^2 + 10kk')\omega/48)), \tag{61}$$

where A_{n,x_0} is a normalization factor which ensures that $\|\psi_n^{(2,x_0)}\| = 1$. For odd values of n the result is nearly identical, i.e.,

$$\psi_x^{(n,2)} = A_{n,x_0} \exp\left(-\left(\frac{1}{2} + (3+2n)\omega/32\right)\xi^2 + (\omega/96)\xi^4\right) \sum_{k=0}^{k'} (h_k^{(n)} \xi^{2k} (1 + (7k - k^2 + 10kk')\omega/48)). \tag{62}$$

The tiny difference $3k \rightarrow 7k$ in the generalized Hermite polynomial is very important, because otherwise the asymptotical convergence $\|\psi_n^{(m,x_0)} - \psi_n^{(x_0)}\| \sim \omega^2$ does not appear. The common exponential part in the third order solution function $\psi_x^{(n,3)}$ reads

$$\exp\left(-\left(\frac{1}{2} + (3+2n)\omega/32 + (53+69n+21n^2)\omega^2/1536\right)\xi^2 + (\omega/96 + (11+6n)\omega^2/1024)\xi^4 - (\omega^2/1280)\xi^6\right). \tag{63}$$

The explicit solution function $\psi_x^{(n,m)}$ solves the asymptotical eigenvalue equation up to the order ω^m and yields a normwise convergence of $\sim \omega^m$.

When employing these asymptotical solutions, one should first study, how accurate eigenvectors are required for the problem at hand. The next step is to choose the order of the solution and the correct truncation with respect to x . Then, the calculations are performed and the results are obtained, hopefully faster than with the conventional approach of numerical diagonalization.⁶

IV. COMMENTS ON SOLVING THE ANSATZ

In this section we discuss how to solve the set of algebraic equations resulting from Eq. (33) as effectively as possible. First we observe that the zeroth order, i.e., terms proportional to arbitrary powers of ξ are satisfied by the fact $\exp(0)=1$. Next, all equations related to terms

$$\{\omega \xi^{n+2-2l'}\}_{l'=0}^{1+k'} \tag{64}$$

are identically satisfied because of the recursion relation (6) rewritten in terms of the coefficients $h_k^{(n)}$. A careful reader notices that terms proportional to $\beta_{k,l=2}^{(n)}$ do appear, but they identically cancel and thus they are not constrained in this order.

From here on, we proceed by recursively solving the coefficients for the next order and also for sufficiently many values of n so that all coefficients in the expansions of $\{\alpha_{kl}^{(n)}\}$ and $\{\beta_{kl}^{(n)}\}$ have been constrained. In reality, we first obtained the solution function $\psi_x^{(n=0,m=6)}$ and a poorly formulated expression for arbitrary second-order solution, i.e., $\psi_x^{(n,m=2)}$, but let us proceed in the way this should be done. Because the equation are quite difficult to handle with pen and paper, we chose to write and simplify the equations with MATHEMATICA software.³³

We first consider the cases $n=0$ and $n=2$ as simple examples. For $n=0$ we expand Eq. (33) up to and including order ω^3 to find

$$\left\{ 1 - \frac{\omega}{2} + \omega^2 \left(\alpha_{1,2}^{(0)} + \frac{1}{8} + \frac{x^2}{2} \right) + \omega^3 \left[-\frac{1}{48} + \frac{\alpha_{1,2}^{(0)}}{2} + \alpha_{1,3} + \alpha_{2,2}^{(0)} + x^2 \left(-\frac{1}{4} - 2\alpha_{1,2}^{(0)} + 6\alpha_{2,2}^{(0)} \right) \right] \right\} = \left[1 - \frac{\omega}{2} + \frac{\omega^2(x^2 + 1/16)}{2} + \frac{\omega^3}{512} \right]. \tag{65}$$

Immediately, we obtain

$$\alpha_{1,2}^{(0)} = -3/32 \quad \text{and} \quad \alpha_{2,2}^{(0)} = 1/96. \tag{66}$$

Inserting these into Eq. (65) gives $\alpha_{1,3}^{(0)} = -53/1536$.

In the case $n=2$ and we examine all terms below the order of ω^4 . The generalized Hermite polynomial now reads

$$H_2^\omega(\xi) = -2 + 4\omega x^2(1 + \beta_{1,1}^{(2)}\omega + \beta_{1,2}^{(2)}\omega^2 + \beta_{1,3}^{(2)}\omega^3) + \mathcal{O}(\omega^5). \tag{67}$$

Expanding all terms and moving them onto the same side yields the equation

$$\begin{aligned} 0 = & \omega^2 \left(\frac{23}{32} + \alpha_{1,2}^{(2)} - 2\beta_{1,1}^{(2)} \right) + \omega^3 \left[-\frac{521}{1536} - \frac{5\alpha_{1,2}^{(2)}}{2} + \alpha_{1,3}^{(2)} + \alpha_{2,2}^{(2)} + \beta_{1,1}^{(2)} - 2\beta_{1,2}^{(2)} \right. \\ & \left. + x^2 \left(-\frac{43}{16} - 12\alpha_{1,2}^{(2)} + 6\alpha_{2,2}^{(2)} \right) \right] + \omega^4 \left[\frac{341}{24576} + \frac{(\alpha_{1,2}^{(2)})^2}{2} - \frac{5\alpha_{1,3}^{(2)}}{2} + \alpha_{1,4}^{(2)} + \alpha_{1,2}^{(2)} \left(\frac{9}{8} - 2\beta_{1,1}^{(2)} \right) \right. \\ & \left. - \frac{\beta_{1,1}^{(2)}}{4} + \beta_{1,2}^{(2)} - 2\beta_{1,3}^{(2)} + x^2 \left(\frac{953}{768} + 2(\alpha_{1,2}^{(2)})^2 - 12\alpha_{1,3}^{(2)} - 37\alpha_{2,2}^{(2)} + 6\alpha_{2,3}^{(2)} + \alpha_{1,2}^{(2)} \left(\frac{29}{2} - 10\beta_{1,1}^{(2)} \right) \right) \right. \\ & \left. - \frac{39\beta_{1,1}^{(2)}}{16} \right] + x^4 \left(\frac{29}{24} + 4\alpha_{1,2}^{(2)} - 32\alpha_{1,3}^{(2)} \right). \tag{68} \end{aligned}$$

Notice that all terms proportional to ω^0 and ω^1 have canceled out, which again shows that the lowest-order approximation for the eigenvalue and eigenstate are already correct and agree with the results for the continuous case. The three coefficients related to the $\psi^{(2,2)}$ can be solved from the coefficients of ω^2 , $\omega^3 x^2$, and $\omega^4 x^4$ and they read

$$\alpha_{1,2}^{(2)} = -7/32, \quad \alpha_{2,2}^{(2)} = 1/96, \quad \text{and} \quad \beta_{1,1}^{(2)} = 1/4. \tag{69}$$

Substituting these into the set of equations and extending the calculation to order ω^6 we find the subsequent coefficients to be

$$\alpha_{1,3}^{(2)} = -275/1536, \quad \alpha_{2,3}^{(2)} = 23/1024, \quad \alpha_{3,3}^{(2)} = -1/1280, \quad \text{and} \quad \beta_{1,2}^{(2)} = 37/256. \tag{70}$$

After solving a sufficient number of coefficients $\alpha_{kl}^{(n)}$ and $\beta_{kl}^{(n)}$ one should start searching for regularities in the solution.

Almost immediately we guessed the polynomial character of $\alpha_{kl}^{(n)}$, first in terms of n and later noticing that they should be written in terms of \hat{n} as in Eq. (44). This considerably helps solving the coefficients $\beta_{kl}^{(n)}$ as for larger values of n the coefficients $\alpha_{kl}^{(n)}$ appear as constants, not unknowns.

In the beginning, we tried to solve all possible terms up to a given order in ω . First one should notice that only terms with $l \leq m$ are required for the solution function $\psi_x^{(n,m)}$. Assuming that the previous orders have been explicitly obtained, means that only the equations corresponding to $m' = m$ in Eq. (34) have to be solved. In addition, generally known coefficients $\alpha_{k,m}^{(n)}$ identically satisfy equations corresponding to the highest powers of ξ . Explicitly, if we assume that coefficients $\{\alpha_{k,m}^{(n)}\}_{k=k_0}^m$ are known, only the equations for

$$\{\omega^m \xi^{n+2k_0-2l'}\}_{l'=0}^{k_0+k'} \tag{71}$$

are required and the expansion of Eq. (33) has to be carried out up to and including the order $\omega^{m+k_0+n/2}$ for coefficients $l \leq m$.

After obtaining a rather complicated expression for the coefficients $\beta_{k,l=3}^{(n)}$, we happened to transform it into form equivalent to the present form and conjecture the general form of $\beta_{kl}^{(n)}$ in Eq. (52). The most important lesson taught by the discretized harmonic oscillator when solving the coefficients is that your numbers may be wrong, but the general forms usually are not. On several occasions, this became painfully obvious when the numbers did not check. Each and every time the general forms were correct, but the used expansion of Eq. (33) or the numbers inserted into it were not.

Later on, we started to study the regularities in the general expressions. The polynomial structure of the coefficients $\alpha_{kl}^{[l']}$ that do not contain any gamma functions was relatively easy obtain, but the other set required a real stroke of luck. We managed to write some of these coefficients $\alpha_{kl}^{[l']}$ as explicit products. After being pointed out, by MATHEMATICA, that the first two could be written in terms of gamma functions, it was only a question of finding the correct gammas before Eq. (45) was written. In order to appreciate the technical part of obtaining the general form of the coefficients we point out that the coefficient $\alpha_{k,k+4}^{(n)}$ was completed by solving the 12th order solution $\psi_{n=0}^{(m=12,x_0)}$ and confirmed by the case $n=1$. Further terms have been obtained by solving the asymptotical differential equations (35).

The regularities in the coefficients $\{\beta_{kl}^{(n)}\}$ have been found out using by studying the expansions with respect to k and k' . By conjecturing the recurring appearance of $(10k'-k)$ in Eqs. (53) and (54) it becomes possible to solve the quantities defined in Eqs. (55)–(58). In addition to these, the general expression for $\rho_{k,l-3}^{(l)}$ can be obtained from the known coefficients.

Finally, we will estimate the difficulty of obtaining the explicit asymptotical solution $\psi_n^{(m,x_0)}$. We assume that both the expansion of the eigenvalues up to the required order and the solution $\psi_n^{(m-1,x_0)}$ have been obtained in advance. The coefficients $\alpha_{km}^{(n)}$ can be determined from the exponential parts of the eigenvectors up to and including the case $n=m-1$. The completely general expressions in Eq. (45) are finished at much slower a pace. The asymptotically satisfied differential equations (35) speed up this process considerably.

Obtaining the coefficients $\beta_{km}^{(n)}$ is more difficult. The general form (52) shows that all states up to $n=4m-3$ must be solved. The explicit expressions for the leading parts, i.e., $B(l,2l-2)$, $B(l,2l-3)$, $\bar{B}(l,2l-3)$, and $\bar{B}(l,2l-4)$ make this task easier by five states. Thus all states up to $n=4l-8$ must be found, unless further general properties are found.

Regardless of these simplifications, the number of required terms and participating equations grows quite fast. Obviously, the general form of the coefficients in the exponential factor is much easier to obtain and thus they should be applied as early as possible. It is also possible that considerable simplifications or generalizations for the known coefficients lurk just around the corner. This has already happened on several occasions so far. We still choose to pause here, as the given general expressions have been validated rather convincingly and it not obvious, how, if at all, the next orders in the expansion would improve the results qualitatively. We hope a solid foundation has been laid for those striving towards the complete, asymptotical solution for the discretized harmonic oscillator.

V. PROVING THE SOLUTION AND SOME GENERAL PROPERTIES

Finally, we attack the difficult problem of actually showing that the solution is a general one. Thus far we have solved the equations for an increasing number of eigenstates using Eq. (33). This formulation is the best if actual numerical values of the coefficients $\alpha_{kl}^{(n)}$ and $\beta_{kl}^{(n)}$ are sought after. This is explained by symbolic math being most effective when the number of unknowns and symbols is as small as possible. In principle, the process explained below could be used for

obtaining recursion relations between the coefficients of the solution and, subsequently, the full solution. Presently, we only show that the equations corresponding to leading orders up to ω^3 are satisfied identically.

We have to solve the equations corresponding to $\{\omega^m \xi^{n+m-2l'}\}_{l'=0}^{k'+m}$ in order to obtain the m th order solution. We have now obtained the explicit solution up to the seventh order so we can check if it is correct. For this purpose, we must write Eq. (33) explicitly in terms of $u := \sqrt{\omega}$ and ξ , although odd powers of u eventually cancel. Multipliers of $\alpha_{kl}^{(n)}$ and $\beta_{kl}^{(n)}$ now read

$$u^{2(l-1)}[(\xi \pm u)^{2k} - \xi^{2k}] \quad \text{and} \quad (\xi \pm u)^{n+2(k-k')}, \tag{72}$$

respectively. On the right-hand side the nontrivial term is given by $\omega \xi^2/2$. Expanding all terms multiplying a fixed term $h_k^{(n)}$ up to the order yields terms

$$\begin{aligned} & h_k^{(n)} \left[1 + u^2 \left(2(k-k') - \hat{n}/2 + \frac{\xi^2}{2} + \frac{(n-1+2(k-k'))(n+2(k-k'))}{\xi^2} \right) \right] \\ & = h_k^{(n)} \left(1 + u^2 \left(-\hat{n}/2 + \frac{\xi^2}{2} \right) \right). \end{aligned} \tag{73}$$

The terms proportional to ξ^2 cancel and equating each power of ξ separately yields an equation

$$h_k^{(n)} 2(k'-k) + [2(k+1)^2 \mp (k+1)] h_{k+1}^{(n)} = 0, \tag{74}$$

where the signs $+$ and $-$ corresponds the even and odd values of n , respectively. The above equation is identically satisfied by the Hermite polynomials, which proves that the first-order solution $\psi_n^{(1,x_0)}$ is correct. A careful observer immediately asks about the second order corrections $\beta_{k,2}^{(n)}$ which also yield terms proportional to u^2 . However, these coefficients are not fixed at all by Eq. (33) in the order u^2 . The only term that is easily solvable from this relation in the dominant coefficient $\alpha_{1,1}^{(n)} = -1/2$ which removes $h_{k-1}^{(n)}$ from the recursion relations. Later on, the dominant coefficients $\{\alpha_{kk}^{(n)}\}_{k=1}^m$ cancel the term $h_{k-m}^{(n)}$ in the order ω^m .

In the next order ω^2 we insert the solved coefficients and obtain for even values of a recursion relation

$$\begin{aligned} & 6(n+2-2k)h_{k-1}^{(n)} + [(2k^3+k^2(42-11n)-6n-3n^2-k(-6+9n-5n^2)]h_k^{(n)} - (1+k)(1+2k) \\ & \times (22+31k+k^2-5n-5kn)h_{k+1}^{(n)} + 2(2k+4)(2k+3)(2k+2)(2k+1)h_{k+2}^{(n)} = 0, \end{aligned} \tag{75}$$

which is again identically satisfied by the Hermite polynomials. For odd values of n , we find a similar recursion relation, once we replace $k' = n/2$ by $k' = (n-1)/2$. This completes the proof in order ω^2 and validates the second-order eigenvectors $\psi_n^{(2,x_0)}$.

In the third order the recursion relation for even values of n reads

$$\begin{aligned} & 180(-4+2k-n)h_{k-2}^{(n)} + 30(62-42k-24k^2+4k^3+74n-10kn-22k^2n+17n^2+10kn^2)h_{k-1}^{(n)} \\ & + [-450n-450n^2-90n^3-10k^5+k^4(-452+105n)+k^3(-2332+2458n-300n^2) \\ & + k^2(4230+4912n-1204n^2+125n^3)+k(-300-585n-1258n^2+179n^3)]h_k^{(n)} \\ & + (1+k)(1+2k)(1022+2705k+3684k^2+326k^3+5k^4-2274n-4430kn-1726k^2n \\ & -50k^3n+454n^2+579kn^2+125k^2n^2)h_{k+1}^{(n)} - 16(1+k)(2+k)(1+2k)(3+2k)(110+101k \\ & +5k^2-50n-25kn)h_{k+2}^{(n)} + 256(1+k)(2+k)(3+k)(1+2k)(3+2k)(5+2k)h_{k+3}^{(n)} = 0. \end{aligned} \tag{76}$$

Because the Hermite polynomials satisfy this and the corresponding relation for odd values of n the solution $\psi_n^{(3,x_0)}$ has been rigorously proven as correct.

The eigenvectors $\psi_n^{(m,x_0)}$ tend to the eigenvectors $\psi_n^{(x_0)}$ of $H(x_0)$ at an asymptotical rate proportional to ω^m . The exact eigenvectors are orthogonal as eigenvectors of a Hermitian matrix and by their closure relation we can write

$$\psi_n^{(m,x_0)} \sim \psi_n^{(x_0)} + \omega^m \sum_{n'} b_{n'} \psi_{n'}^{(x_0)}, \tag{77}$$

where $b_{n'}$ are finite constants such that $\sum_n |b_n|^2 < \infty$ in the limit $\omega \rightarrow 0$. The orthonormality relation for the asymptotical solutions thus reads

$$\langle \psi_n^{(m,x_0)} | \psi_{n'}^{(m,x_0)} \rangle = \delta_{nn'} + \mathcal{O}(\omega^m), \tag{78}$$

provided that the sum $\sum_n |b_n|$ is finite for both states. In other words, the eigenvectors $\psi_n^{(m,x_0)}$ become orthonormal at the rate of ω^m . Numerical checks seem to confirm this, at least for relatively small values of n .

As a final effort, we outline a plausible “proof” for the asymptotical convergence. As a first step, we show that without loss of generality we can examine a finite truncation of the eigenvector $\psi_n^{(n,x_0)}$, the vector $\psi_{n,j_0}^{(x_0)} := \{\psi_{j-x_0}^{(n)}\}_{j=-j_0}^{j_0}$ for sufficiently large j_0 . For sufficiently large values of $|j|$ the eigenvalue λ_n becomes insignificant in Eq. (32) and we write an approximate equation

$$(\psi_{x-1}^{(n)} - 2\psi_x^{(n)} + \psi_{x+1}^{(n)})/(x^2) = \omega^2. \tag{79}$$

For sufficiently large values of x and/or j the sign of $\psi_x^{(n)}$ is constant and this equation shows that the function $\tilde{\psi}_x = \psi_{j_0-x_0}^{(n)} \exp(-\omega(x^2 - j_0^2)/(2 + \epsilon))$, for some small $\epsilon > 0$, is a dominant sequence for $\psi_x^{(n)}$. Now, the limiting sequence of norms

$$\lim_{j_0 \rightarrow \infty} \|\psi_n^{(x_0)} - \psi_{n,j_0}^{(x_0)}\| \tag{80}$$

vanishes exponentially with respect to j_0 . In other words, we can always find a finite j_0 such that the error in the norm is sufficiently small.

Next we use the fact that solution $\psi_n^{(m,x_0)}$ satisfies the eigenvalue equation (32) up to the order ω^m when written in terms of ξ . Thus we can write

$$\frac{\psi_{x-1}^{(n,m)} + \psi_{x+1}^{(n,m)}}{2\psi_x^{(n,m)}(-\lambda_n + \omega\xi^2/2)} = 1 + \mathcal{O}(\omega^{m+1}). \tag{81}$$

We fix the scales of the eigenvectors by setting $(\psi_n^{(m,x_0)})_j = (\psi_n^{(x_0)})_j$ for an arbitrary j . It would be very tempting to say that Eq. (81) implies $(\psi_n^{(m,x_0)})_{j+1} = (\psi_n^{(x_0)})_{j+1}(1 + \mathcal{O}(\omega^{m+1}))$ and then wonder why convergence is not asymptotically proportional to ω^{m+1} . As already explained the solution $\psi_n^{(m,x_0)}$ does not fix the coefficients $\beta_{k,m+1}^{(n)}$ which most definitely yield terms proportional to ω^m . Thus we obtain a relation

$$(\psi_n^{(m,x_0)})_{j+1} = (\psi_n^{(x_0)})_{j+1}(1 + \mathcal{O}(\omega^m)). \tag{82}$$

By matching the eigenvectors at $j=0$, and expanding the components to the finite values $\pm j_0$ shows that the order of error is ω^m for all components with $|j| \leq j_0$. Because the error caused by the truncation is insignificant the result holds for the full eigenvectors and we obtain the desired result

$$\|\psi_n^{(m,x_0)} - \psi_n^{(x_0)}\| \sim \omega^m, \quad (83)$$

or at least show that the result is quite plausible.

VI. CONCLUSIONS

We have obtained an explicit, asymptotical solution for the discretized harmonic oscillator. Both the eigenvalues and eigenvectors have been obtained and we can choose a prespecified rate of convergence towards the exact solutions. This is done by truncating the ansatz solution accordingly. Because the problem can be mapped onto the Mathieu differential equation, we simultaneously provide asymptotical expressions for the Mathieu functions. The Schrödinger equation of the quantum pendulum corresponds to the Mathieu equation, which yields immediate applications for the results.

The method described above can be generalized to accommodate several coordinate dimensions with only minor changes. This should make the results of Ref. 29 both more transparent and more rigorous. The tunnelling–charging Hamiltonian of a Cooper pair pump corresponds to a modified multidimensional Mathieu equation.

Alternatively, ansatzes similar to Eq. (30) could be constructed in case of difference equations that become identical to analytically solvable differential equations in some asymptotical limit. Initially, the problem assumes the form of an infinite-dimensional, two-parameter (eigenvalue) problem, where the asymptotical solutions (eigenvalues and eigenvectors) must be obtained. The ansatz maps the problem onto an infinite set of algebraic equations that must be solved. If the form of the ansatz is correct, one may determine some general properties of the exact solution.

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³³Many MATHEMATICA notebooks containing much of the data used in calculations is available at <http://www.cc.jyu.fi/~mimaau/harmonic>. The general results have been compiled into Mathieunewgen.nb, where the eigenvalue is given up to 19th order as well as the general expression for the coefficient $\alpha_{k,k+7}^{(n)}$.