tion, but is now called Snel's law (or, wrongly, Snell's law) after its first discoverer, from whom Descartes has even been alleged to have plagiarized it. But the case is different for mathematics, where the Cartesian revolution was most profound and has been long lasting. We honor one of Descartes's discoveries in algebra with the name Descartes's law of signs. By calling the system of rectangular coordinates Cartesian coordinates, mathematicians continue to celebrate Descartes as the author of a great revolution at the beginning of modern science.[2]

10 The Newtonian Revolution

The Newtonian revolution differs from those other revolutions (actual or alleged) in science and in mathematics which we have been considering in that Newton was said in his own lifetime to have created a revolution. He was recognized by his contemporaries for the revolution of the calculus and for a revolution in the science of mechanics created by his *Philosophiae Naturalis Principia Mathematica.* From a historical vantage point, Newton was an extraordinary figure because he made so many fundamental contributions to different fields: pure and applied mathematics; optics and the theory of light and colors; design of scientific instruments; codification of dynamics and formulation of the basic concepts of this subject; invention of the primary concept of physical science (mass); invention of the concept and law of universal gravity and its elaboration into a new system of the universe gravity; invention of the gravitational theory of tides; and formulation of the new methodology of science. He also worked on heat, the chemistry and theory of matter, alchemy, chronology, interpretation of Scripture, and other topics. The range of his intellectual career never ceases to astonish.

The Newtonian revolution in mathematics had two aspects: the invention of the calculus (an honor he shares with Leibniz) and the application of mathematics to physics and astronomy. It was the latter which produced the Newtonian revolution in science (as opposed to a revolution in mathematics). Of course, Newton had great predecessors in the art of developing natural philosophy by mathematical principles: Stevin, Galileo, Kepler, Wallis, Hooke, Huygens. In this sense the Newtonian revolution in science was the culmination of a multiauthored effort, going back to the beginning of the Scientific Revolution, rather than the creation by Newton of something wholly new. Yet the simplest comparison of Newton's *Principia* with Kepler's *Astronomia Nova,* Galileo's *Two New Sciences,* Wallis's *Mechanics,* Hooke's writings on motion, or the treatment of accelerated motions in Huygens's treatise on the pendulum clock shows a difference of several orders of magnitude in depth, scope, and technique. It is because of the size of this quantum jump that Newton's *Principia* is the "epoch" (as Clairaut said in 1747) of a "revolution in physical science."

It is sometimes alleged that Newton created a synthesis, presumably putting together disparate ideas or principles of such scientists as Kepler, Galileo, or Hooke. But Newton's revolutionary science was hardly a melding or assembling of such ideas or principles, since in actual fact Newton's *Principia* declared their falsity. Surely a 'true' science cannot result from a mere amalgamation of false ideas and principles. Among such notions whose falsity is exhibited by Newton in the *Principia* are the following:

Kepler: the three planetary laws are "true" descriptions of the motion of the planets; a solar force exerted on those bodies diminishes directly as the distance and acts only in or near the plane of the ecliptic; the sun must be a huge magnet; because of its "natural inertia," a moving body will come to rest whenever the motive force ceases to act.

Descartes: the planets are carried around by a sea of aether moving in huge vortices; atoms do not (cannot) exist, and there is no vacuum or void space.

Galileo: the acceleration of bodies falling toward the earth is constant at all distances, even as far out as the moon; the moon cannot possibly have any influence on (or be the cause of) the tides in the sea.

Hooke: the centripetal inverse-square force acting on a body (with a component of inertial motion) produces orbital motion with a speed inversely proportional to the distance from the center of force: this speed law is consistent with Kepler's area law.

We may further observe that Newton also denied the existence of 'centrifugal' forces, which were basic to the development of Huygens's physics of motion. In their place Newton introduced a concept of 'centripetal' force, a name he chose because it was similar — though opposite in sense or direction — to Huygens's 'vis centrifuga'.

Comparison and contrast of Newton's *Principles of Philosophy* (the name he often used to refer to his book) and Descartes's *Principles of Philosophy* show the nature of the Newtonian revolution. For the critical reader one of the extraordinary aspects of Descartes's *Principles* is that it is devoid of mathematics, being largely devoted to philosophy and to philosophical principles of physics or natural philosophy. Only two of the four Parts deal with physics proper and the development of the cosmic system of vortices. Here Descartes does set forth the quantitative rules for impact which we have seen to be wrong in each example. Descartes included these rules as a subset of his third law of nature. But when Wallis published the true rules in the *Philosophical Transactions of the Royal Society,*[1] they bore the more restricted and more correct title of "Laws of Motion." Newton began his *Principles of Philosophy* with a set of "definitions" followed by "axioms or laws of motion," of which the first two correspond roughly to Descartes's first two laws of nature. Newton seems to have transformed the Cartesian "regulae quaedam sive leges naturae" into his own "axiomata sive leges motus." Newton's three laws of motion, the axioms to which he reduced the system of rational mechanics, were: (1) the principle of inertia, that a body will persevere in its state of rest or of uniform motion straight forward unless acted on by an external force; (2) the relation of a force to its dynamical effect, that an impulsive (or continuous) external force produces a change (change in a unit time for a continuous force) in the momentum of a body in the direction of action of the force; (3) the equality of action and reaction.

Newton also transformed Descartes's title of *Principia Philosophiae* into *Philosophiae naturalis Principia mathematica,* thus boasting that in mathematicizing the principles he had constructed a natural philosophy rather than a general philosophy. Newton's *Principia* is not only mathematical in the development of the principles and in the proofs and applications of the propositions; it also sets forth a significant new mode of using mathematics in natural philosophy.

Newton's *Principia* is a remarkable book on many levels. It contains original results in pure mathematics (theory of limits and geometry of conic sections), it develops the primary concepts of dynamics (mass, momentum, force), it codifies the principles of dynamics (three laws of motion), and it shows the dynamical significance of Kepler's three laws of planetary motion and of Galileo's experimental conclusion that bodies with unequal weights will fall freely (at the same place on earth) with identical accelerations and speeds. It develops the laws of curved motions, the analysis of pendulums, and the nature of motions constrained to surfaces, and it shows how to deal with the motion of particles in continually varying force fields. Newton also indicates the way to analyze wave motions, and he explores the manner in which bodies move in various resisting mediums. The crown of all appears in the final book 3, in which he discloses the Newtonian system of the universe—regulated by gravity, by the action of a general force, of which one particular manifestation is the familiar terrestrial weight. Here Newton treats at length of the orbits of planets and their satellites, the motions and paths of comets, and the production of tides in the sea.

As an example of the new level of thought in the *Principia,* consider the motion of the moon with its apparent irregularities. For a millennium and a half, astronomers had dealt with the moon's motion by constructing geometric schemes without reference to cause. Now, Newton showed that the chief source of the 'lunar inequalities' was the phenomenon of perturbation, chiefly the result of the gravitational action of the sun as well as of the earth on the moon. With the publication of the *Principia* in 1687, it became possible to deal with this problem by starting from first principles or causes and then studying the effects. As a reviewer of the second edition of the *Principia* observed, this was entirely a new way to deal with the problem.[2]

Perhaps the greatest triumph of all was the explanation that tides are caused by the gravitational pull of the sun and moon on the seas. "The ebb and flow of the sea," Newton declared (in bk. 3, prop. 24), "arise from the actions of the sun and moon." The magnitude of his achievement is shown by his prediction of the oblate shape of the earth on the basis of his analysis of precession and the nonsymmetrical pull of the moon on the earth's supposed equatorial bulge.

Some analysts would see the greatness of the *Principia* expressed in the commitment to an inertial physics; for Newton inertia is a property of mass. Newton is the first writer to make a clear distinction between mass and weight and to recognize, furthermore, that a body's mass has two separate and distinct aspects. Mass is a measure of the body's resistance to being accelerated or undergoing a change in its state of motion or of rest; this is its inertia. (Newton sometimes used the term 'force of inertia' or 'vis inertiae'—but this type of force differs from forces that are 'active' and that can produce accelerations.) But a body's mass is also a measure of the body's response to a given gravitational field. But why should there be a relation between a body's (inertial) resistance to acceleration and its (gravitational) response to a gravitational field? In classical physics there is no reason. Newton had the insight to recognize that this relation must rest on the foundation of experiment, and so he proceeded to prove by experiment this constancy between inertia and gravity. It is only in Einstein's relativity theory that there is a logical necessity for this equivalence of 'inertial' mass and 'gravitational' mass. Einstein greatly admired Newton for having had so deep an insight into this problem and for having recognized that the only Newtonian grounds for this equivalence were experimental.

The nature of the mathematics in Newton's *Principia* is often misunderstood. A superficial turning of the pages gives the impression that the mathematics used by Newton is geometry, particularly Greek geometry. The style seems to be that of Euclid or Apollonius. But a closer examination shows that Newton is developing the subject by the calculus, by stating relations geometrically in ratios and proportions and at once considering the 'limit' as a fundamental quantity vanishes (or is nascent). Hence, although Newton does not develop an algorithm of the calculus (or 'fluxions') which he then applies systematically, he does make extensive use of limiting procedures which are clearly equivalent to using the calculus or which can readily be translated into the symbolism of either the Newtonian or the Leibnizian algorithm. Recognizing this aspect of the *Principia,* the Marquis de l'Hôpital observed (as Newton proudly noted) that the mathematics of the book is almost entirely the calculus. This would be further evident to any careful reader from the development of the theory of limits in section 1 of book 1 and from the explicit theory of fluxions (the Newtonian version of the differential calculus) in section 2 of book 2. Additionally, the *Principia* was notable for other original uses of mathematics such as the extensive use of infinite series.

Newton's Style

The essence of Newton's revolutionary science, as I see it, is to be found in what I have called the 'Newtonian style'. This can be seen most easily in Newton's treatment of Kepler's laws in the *Principia.*[3] Newton begins with a purely mathematical construct or imagined system — not merely a case of nature simplified but a wholly invented system of the sort that does not exist in the real world at all. Here by 'real' world is meant only the external world as revealed by experiment and observation. In this system or construct, a single mass-point moves about a center of force. Newton shows by mathematics (bk. 1, prop. 1) that if in this construct or system a force is constantly directed from the orbiting mass point or particle to the immobile center of force, then Kepler's law of areas (his second law) will hold. He next proves the converse (prop. 3), that if the law of areas holds there must be such a centripetal or centrally directed force. Hence the existence of a centripetal force is proved to be both a necessary and sufficient condition for Kepler's law of areas. Then Newton shows that if the orbit is an ellipse, the central force must vary inversely as the square of the distance. Finally he proves that if under such a condition of force there are several orbiting mass points, which do not interact with each other — or (what comes to the same thing) if the motion of any given mass point is compared with what its motion would be at a somewhat different distance from the center — then Kepler's third or harmonic law will hold. Incidentally, we may observe that Newton has shown here for the first time the dynamical significance of each of Kepler's laws. Newton's procedure thus far constitutes a purely mathematical phase one.

In phase two, Newton compares his mental construct with the real world. At once, of course, he discovers that in the real world (for instance, in our solar system), orbiting bodies do not move about 'mathematical' centers of force but about other real bodies. The moon moves around the earth; the earth and the other planets move around the sun. Accordingly, in order to bring his mental construct or imagined system more into harmony with the real world, Newton modifies the system so that there are now two mass points. One is at the center and attracts the one which is moving in orbit, constantly drawing it away from its otherwise rectilinear inertial path. But according to the principle that to every action there must be an equal and opposite reaction (Newton's third law of motion), it follows that if the central body attracts the orbiting body, then the orbiting body must also attract the central body. Hence the mental construct becomes enlarged to a system of two interacting bodies. Newton proceeds to show that under these circumstances the orbiting body does not any longer move in a simple ellipse around the central body at a focus; rather, he finds that both will move in ellipses around their common center of gravity.

This two-body system constitutes a modified phase one in which Newton once again develops mathematically the properties of his (now revised) mental construct. He next compares the modified system with the external world, a modified phase two. Of course, he finds that this system also does not conform to the real world around us. For instance, in our solar system there is not just a single planet moving around the sun but several. Accordingly, to make his mental construct conform more closely to the system of the external world, Newton moves onto yet another phase one. He introduces two or more mass points orbiting about the central mass point, not just one. It follows, again as a result of the application of Newton's third law, that each of these orbiting mass points both is attracted by the central body and attracts it. In other words, a consequence is that each orbiting mass point is both a body that can be attracted and a center of attractive force. Each of these orbiting bodies will act upon and be acted upon by every other orbiting body. The system contains bodies which act by perturbations on one another, and these perturbations produce a slight departure from Kepler's laws. Newton then proceeds to find the quantitative measure of the deviation from Kepler's laws in our solar system.

In this kind of contrapuntal alternation between mathematical constructs and comparisons with the real world, between a phase one and a phase two, Newton advances from a one-body system not only to a many-body system but also to a system of orbiting bodies which have satellites, such as the moons of the earth, Saturn, and Jupiter. Thus far he has been considering mass points rather than physical bodies, because he has not yet introduced considerations of size and shape, but eventually he shifts the level of discussion from mass points to physical bodies with significant dimensions and figures.

The progression I have described is not merely a twentieth-century after-the-fact analysis of the way Newton presents his subject in the *Principia*. It also corresponds to the documented stages of development of Newton's ideas.[4] In the autumn of 1684 Newton wrote a tract (*De Motu*) in which he presented the results of his study of Kepler's laws and other aspects of the subject. There he shows that a central force is a necessary and sufficient condition for the law of areas, and that an elliptical orbit implies that the force varies as the inverse square of the distance, much as in the later *Principia*. But he has not as yet recognized that his proofs apply only to a mental construct of a one-body system and so he proudly writes: "Scholium: Therefore the major planets revolve in ellipses having a focus in the center of the sun and by radii drawn [from the planets] to the sun describe areas proportional to the times, entirely as Kepler supposed." Before long Newton realized that the planets cannot in fact move in simple Keplerian elliptical orbits. He saw that his results apply only to an artificial one-body system in which the earth is reduced to a mass point and the sun to an immobile center of force.

In December 1684 Newton completed a revised draft of *De Motu* that describes planetary motion in the context of an interactive many-body system. Unlike the earlier draft, the revised one concludes that "the planets neither move exactly in ellipses nor revolve twice in the same orbit." This conclusion led Newton to the following result:

> There are as many orbits to each planet as it has revolutions, as in the motion of the Moon, and each orbit depends on the combined motions of all the planets, not to mention the actions of all these on one another . . . To consider simultaneously the causes of so many motions and to define the motions themselves by exact laws allowing of convenient calculation exceeds, unless I am mistaken, the power of the entire human intellect.

Newton had come to perceive that the planets act gravitationally on one another. The passage cited above expresses this perception in unambiguous language: "eorum omnium actiones in se invicem" (the actions of all of them on one another). A consequence of this mutual gravitational attraction is that all three of Kepler's laws are not strictly true in the world of physics but are true only for a mathematical construct in which masses that do not interact with one another orbit either a mathematical center of force or a stationary attracting body. The distinction Newton draws between the realm of mathematics, in which Kepler's laws are truly laws, and the realm of physics, in which they are only "hypotheses" (or approximations), is one of the revolutionary features of Newtonian celestial dynamics.

In an early draft of what was to become book 3 of the *Principia*, Newton showed how considerations of the third law of motion led to the concept of a mutual force between the sun and each planet, between a planet and its satellites, and between any two planets. The same considerations lead to the revolutionary new idea that any and all bodies in the universe must "attract one another." He proudly presented this conclusion with the explanatory comment that in any pair of terrestrial bodies the magnitude of the attractive force is so small that it is unobservable. "It is possible," he wrote, "to observe these forces only in the huge bodies of the planets." Of all the planets, Jupiter and Saturn are the most massive, and so he sought orbital perturbations in their motions. With the help of John Flamsteed, Newton found that the orbital motion of Saturn is indeed perturbed when the two planets are closest together.

In book 3 of the *Principia*, which is concerned with the system of the world but is somewhat more mathematical than the earlier version, Newton treats the topic of gravitation in essentially the same way. First, in what is called the moon test, he extends the weight force, or terrestrial gravity, to the moon and demonstrates that the force varies inversely with the square of the distance. Then he identifies the same terrestrial force with the force of the sun on the planets and the force of a planet on its satellites. All these forces he now calls gravity. With the aid of the third law of motion he transforms the concept of a solar force on the planets into the concept of a mutual force between the sun and the planets. Similarly, he transforms the concept of a planetary force on the satellites into the concept of a mutual force between planets and their satellites and between satellites. The final transformation is the notion that all bodies interact gravitationally.

My analysis of the stages of Newton's thinking should not be taken as diminishing the extraordinary force of his creative genius; rather, it should make that genius plausible. The analysis shows Newton's fecund way of thinking about physics, in which mathematics is applied to the external world as it is revealed by experiment and critical observation. Because he did not assume that the construct is an exact representation of the physical universe, he was free to explore the properties and effects of a mathematical attractive force even though he found the concept of a grasping force "acting at a distance" to be abhorrent and not admissible in the realm of good physics. Next he compared the consequences of his mathematical construct with the observed principles and laws of the external world such as Kepler's law of areas and law of elliptical orbits. Where the mathematical construct fell short Newton modified it. This way of thinking, which I call the Newtonian style, is captured by the title of Newton's great work: *Mathematical Principles of Natural Philosophy.*

The law of universal gravitation explains why the planets follow Kepler's laws approximately and why they depart from the laws in the way they do. It was the law of universal gravitation which demonstrated why (in the absence of friction) all bodies fall at the same rate at any given place on the earth and why the rate varies with elevation and latitude. The law of gravitation also explains the regular and irregular motions of the moon, provides a physical basis for understanding and predicting tidal phenomena, and shows how the earth's rate of precession, which had long been observed but not explained, is the effect of the moon's pulling on the earth's equatorial bulge. Since the mathematical force of attraction works well in explaining and predicting the observed phenomena of the world, Newton decided that the force must "truly exist" even though the received philosophy to which he adhered did not and could not allow such a force to be part of a system of nature. And so he called for an inquiry into how the effects of universal gravity might arise.

Although Newton at times thought universal gravity might be caused by the impulses of a stream of aether particles bombarding an object or by variations in an all-pervading aether, he did not advance either of these notions in the *Principia* because, as he ultimately said, he would "not feign hypotheses" as physical explanations. The Newtonian style had led him to a mathematical concept of universal force, and that style led him to apply his mathematical result to the physical world even though it was not the kind of force in which he could believe.

Some of Newton's contemporaries were so troubled by the idea of an attractive force acting at a distance that they could not begin to explore its properties, and they found it difficult to accept the Newtonian physics. They could not go along with Newton when he said he had not been able to explain how gravity works but that "it is enough that gravity really exists and suffices to explain the phenomena of the heavens and the tides." Those who accepted the Newtonian style fleshed out the law of universal gravity, showed how it explains many other physical phenomena, and demanded that an explanation be sought of how such a force could be transmitted over vast distances through apparently empty space. The Newtonian style enabled Newton to study universal gravity without premature inhibitions that would have blocked his great discovery. The eighteenth-century biologist Georges Louis Leclerc de Buffon once wrote that a man's style cannot be distinguished from the man himself. In the case of Newton his greatest discovery cannot be separated from his style.

Acceptance of a Newtonian Revolution

There are numerous testimonials to the Newtonian revolution in science. The eighteenth-century historian of science Jean-Sylvain Bailly wrote that "Newton overturned or changed all ideas": his "philosophy brought about a revolution." Bailly was not content merely to state generalities concerning the Newtonian revolution in science. As he saw it, the key that in Newton's hands unlocked the celestial mysteries was mathematics: geometry. As Bailly put it: "What is supposed to make things move is what really makes things move; the demonstration was complete. Newton alone, with his mathematics [géométrie], divined the secret of nature."

With rare insight, Bailly saw that "the advantage of mathematical solutions is that they are general." The argument that if the planets move according to Kepler's laws, they must be "impelled by a force residing in the sun" depends only on mathematical or geometrical considerations and general principles of motion. No special physical properties of the sun appear in Newton's argument, which differs from Kepler's in that the latter had invoked such special qualities of the sun as its magnetic force and the orientation of its poles. Accordingly, the identical mathematical argument shows that the satellites of Jupiter and Saturn, subject to the same laws of Kepler, must be equally "impelled by forces residing in these two planets." In other words, Jupiter and Saturn are to their satellite systems what the sun is to the planetary system, the only difference being one of extent and power. And the same is true of the earth and our moon (Bailly 1785, vol. 2, bk. 12, sec. 9, pp. 486f.).

Bailly himself was willing to accept the concept and principle of a universal gravitating force, since so many phenomena were explained by its use: so many of the observed data and experiential laws could be derived by mathematics from the properties of universal gravity (sec. 41, pp. 555f.). He was aware, however, that at first many scientists (notably in France) made a distinction between the Newtonian system as mathematical and as a true natural philosophy. Thus with respect to Maupertuis, who (according to Bailly) "appears to us to have been . . . the first of our mathematicians to have used the principle of attraction," Bailly (vol. 3 ("discours premier"): 7) had to point out that "at first he considered it only in relation to its calculable effects; he accepted gravitation as a mathematician, but not as a physicist." That is, Maupertuis went along with the Newtonian mathematical system or construct (our phases one and two) but would not grant that in the system of the world (phase three) Newton was necessarily dealing with reality.

In fact, in a paper "On the Laws of Attraction" (1732), Maupertuis had been very explicit on this point. "I do not at all consider," he wrote, "whether Attraction accords with or is contrary to sound Philosophy."